# CS738: Advanced Compiler Optimizations The Untyped Lambda Calculus

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#### **Reference Book**

#### Types and Programming Languages by Benjamin C. Pierce



#### t := x – Variable

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t := 
$$x$$
 - Variable  
|  $\lambda x.t$  - Abstraction

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Parenthesis, (...), can be used for grouping and scoping.

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λx.t<sub>1</sub>t<sub>2</sub>t<sub>3</sub> is an abbreviation for λx.(t<sub>1</sub>t<sub>2</sub>t<sub>3</sub>), i.e., the scope of x is as far to the right as possible until it is

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- λxyz.t is an abbreviation for λxλyλz.t which in turn is abbreviation for λx.(λy.(λz.t)).

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- Renaming a λ variable, including all its bound occurrences, does not change the meaning of an expression. For example, λx.x x y is equivalent to λu.u u y

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- But it is not same as  $\lambda x.x x w$
- Can not change free variables!

 $\beta$ -reduction (Execution Semantics)

► if an abstraction \u03c6 x.t<sub>1</sub> is applied to a term t<sub>2</sub> then the result of the application is

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  - the body of the abstraction t<sub>1</sub> with all free occurrences of the formal parameter x replaced with t<sub>2</sub>.
- For example,

$$(\lambda f \lambda x.f(f x)) g \xrightarrow{\beta} \lambda x.g(g x)$$

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### Caution

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- The following reduction is WRONG

$$(\lambda x \lambda y. x)(\lambda x. y) \xrightarrow{\beta} \lambda y. \lambda x. y$$

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- During β-reduction, make sure a free variable is not captured inadvertently.
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$$(\lambda x \lambda y.x)(\lambda x.y) \xrightarrow{\beta} \lambda y.\lambda x.y$$

• Use  $\alpha$ -renaming to avoid variable capture

$$(\lambda x \lambda y. x)(\lambda x. y) \stackrel{\alpha}{\longrightarrow} (\lambda u \lambda v. u)(\lambda x. y) \stackrel{\beta}{\longrightarrow} \lambda v. \lambda x. y$$

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#### Exercise

Apply β-reduction as far as possible
(λx y z. x z (y z)) (λx y. x) (λy.y)
(λx. x x)(λx. x x)
(λx y z. x z (y z)) (λx y. x) ((λx. x x)(λx. x x))

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• Multiple ways to apply  $\beta$ -reduction



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Some may not terminate

- Multiple ways to apply β-reduction
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- However, if two different reduction sequences terminate then they always terminate in the same term

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Also called the Diamond Property

- Multiple ways to apply β-reduction
- Some may not terminate
- However, if two different reduction sequences terminate then they always terminate in the same term
  - Also called the Diamond Property
- Leftmost, outermost reduction will find the normal form if it exists

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Where is the other stuff?



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Constants?

Where is the other stuff?

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#### Constants?

Numbers

Where is the other stuff?

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#### Constants?

- Numbers
- Booleans

Where is the other stuff?

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#### Constants?

- Numbers
- Booleans
- Complex Types?

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- Constants?
  - Numbers
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- Complex Types?
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- Constants?
  - Numbers
  - Booleans
- Complex Types?
  - Lists
  - Arrays

Where is the other stuff?

- Constants?
  - Numbers
  - Booleans
- Complex Types?
  - Lists
  - Arrays
- Don't we need data?

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# Programming in $\lambda$ Calculus

Where is the other stuff?

- Constants?
  - Numbers
  - Booleans
- Complex Types?
  - Lists
  - Arrays
- Don't we need data?

Abstractions act as functions as well as data!

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▶ We need a "Zero"



▶ We need a "Zero"

"Absence of item"



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And something to count

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- And something to count
  - "Presence of item"

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- Intuition: Whiteboard and Marker

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  - However, other pairs of objects will work as well

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Lets translate this intuition into λ-expressions

#### **>** Zero = $\lambda m w. w$



#### ► Zero = \lambda m w. w

No mark on the whiteboard



Zero = λm w. w
No mark on the whiteboard
One = λm w. m w

Zero = λm w. w

No mark on the whiteboard

- One = λm w. m w
  - One mark on the whiteboard

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▶ ...

Zero = λm w. w

No mark on the whiteboard

► One = \(\lambda\) m w. m w

One mark on the whiteboard

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Two =  $\lambda m w. m (m w)$ 

Zero = λm w. w

No mark on the whiteboard

One = λm w. m w

One mark on the whiteboard

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- Two =  $\lambda m w. m (m w)$
- ▶ ...
- What about operations?

Zero = λm w. w

No mark on the whiteboard

One = λm w. m w

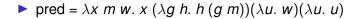
One mark on the whiteboard

- Two = λm w. m (m w)
- ▶ ...
- What about operations?

add, multiply, subtract, divide, ...?

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Succ = 
$$\lambda x m w. m (x m w)$$





# pred = λx m w. x (λg h. h (g m))(λu. w)(λu. u) Verify: pred N = N - 1

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Verify: nminus M N = max(0, M - N) – natural subtraction

True and False



- True and False
- Intuition: Selection of one out of two (complementary) choices

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True = 
$$\lambda x y$$
. x

- True and False
- Intuition: Selection of one out of two (complementary) choices

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True = 
$$\lambda x y$$
. x

False =  $\lambda x y$ . y

- True and False
- Intuition: Selection of one out of two (complementary) choices

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- True =  $\lambda x y$ . x
- False =  $\lambda x y$ . y
- Predicate:

- True and False
- Intuition: Selection of one out of two (complementary) choices

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- True =  $\lambda x y$ . x
- False =  $\lambda x y$ . y
- Predicate:
  - isZero =  $\lambda x$ . x ( $\lambda u$ .False) True

## **Operations on Booleans**

#### Logical operations

and = 
$$\lambda p q. p q p$$
  
or =  $\lambda p q. p p q$   
not =  $\lambda p t f. p f t$ 

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# **Operations on Booleans**

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► The conditional operator if

$$if = \lambda c e_t e_f. (c e_t e_f)$$

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# **Operations on Booleans**

#### Logical operations

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or =  $\lambda p q. p p q$   
not =  $\lambda p t f. p f t$ 

► The conditional operator if

if c e<sub>t</sub> e<sub>f</sub> reduces to e<sub>t</sub> if c is True, and to e<sub>f</sub> if c is False

if = 
$$\lambda c e_t e_f (c e_t e_f)$$

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#### More such types can be found at

https://en.wikipedia.org/wiki/Church\_encoding



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It is fun to come up with your own definitions for constants and operations over different types

# More...

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- It is fun to come up with your own definitions for constants and operations over different types
- or to develop understanding for existing definitions.

# We are missing something!!

The machinery described so far does not allow us to define Recursive functions

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- Factorial, Fibonacci, ...
- There is no concept of "named" functions
  - So no way to refer to a function "recursively"!
- Fix-point computation comes to rescue

# Fix-point and Y-combinator

#### A fix-point of a function f is a value p such that f p = p

# Fix-point and Y-combinator

- A fix-point of a function f is a value p such that f p = p
- Assume existence of a magic expression, called
   Y-combinator, that when applied to a λ-expression, gives its fixed point

$$Y f = f (Y f)$$

# Fix-point and Y-combinator

- A fix-point of a function f is a value p such that f p = p
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$$Y f = f (Y f)$$

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Y-combinator gives us a way to apply a function recursively

## **Recursion Example: Factorial**

fact =  $\lambda n$ . if (isZero n) One (mult n (fact (pred n))) =  $(\lambda f n$ . if (isZero n) One (mult n (f (pred n)))) fact

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# **Recursion Example: Factorial**

- fact =  $\lambda n$ . *if* (isZero *n*) One (mult *n* (fact (pred *n*))) = ( $\lambda f$  *n*. *if* (isZero *n*) One (mult *n* (*f* (pred *n*)))) fact
- fact = g fact

#### fact is a fixed point of the function

 $g = (\lambda f n. if (isZero n)One (mult n (f (pred n))))$ 

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# **Recursion Example: Factorial**

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- fact = g fact

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 $g = (\lambda f n. if (isZero n)One (mult n (f (pred n))))$ 

Using Y-combinator,

fact = 
$$Y g$$

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fact 2 = 
$$(Y g) 2$$



fact 2 = 
$$(Y g) 2$$
  
=  $g(Y g) 2$  - by definition of Y-combinator

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# fact 2 = (Y g) 2= g (Y g) 2 - by definition of Y-combinator = $(\lambda fn. if (isZero n) 1 (mult n (f (pred n)))) (Y g) 2$

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#### fact 2 = (Y g) 2

- = g(Yg) 2 by definition of Y-combinator
- =  $(\lambda fn. if (isZero n) 1 (mult n (f (pred n)))) (Y g) 2$

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- =  $(\lambda n. if (isZero n) 1 (mult n ((Y g) (pred n)))) 2$
- = if (isZero 2) 1 (mult 2 ((Y g)(pred2)))

## fact 2 = (Y g) 2

- = g(Yg) 2 by definition of Y-combinator
- =  $(\lambda fn. if (isZero n) 1 (mult n (f (pred n)))) (Y g) 2$

- =  $(\lambda n. if (isZero n) 1 (mult n ((Y g) (pred n)))) 2$
- = if (isZero 2) 1 (mult 2 ((Y g)(pred2)))
- = (mult 2 ((Y g) 1))

## fact 2 = (Y g) 2

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. . .

### = (mult 2 (mult 1 (*if* (isZero 0) 1 (...))))

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- = (mult 2 (mult 1 (if (isZero 0) 1 (...))))
- = (mult 2 (mult 1 1))

= 2

 Y-combinator allows to unroll the body of loop once—similar to one unfolding of recursive call

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- Sequence of Y-combinator applications allow complete unfolding of recursive calls

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BUT, what about the existence of Y-combinator?

Many candidates exist

$$Y_1 = \lambda f. (\lambda x. f (x x)) (\lambda x. f (x x))$$

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Many candidates exist

$$Y_1 = \lambda f. (\lambda x. f (x x)) (\lambda x. f (x x))$$

 $Y = \lambda abcdefghijklmnopqstuvwxwzr.r(thisisafixedpointcombinator)$ 

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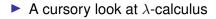
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• Verify that (Y f) = f (Y f) for each







# Summary

- A cursory look at  $\lambda$ -calculus
- Functions are data, and Data are functions!

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# Summary

- A cursory look at λ-calculus
- Functions are data, and Data are functions!
- Not covered but important to know: The power of λ calculus is equivalent to that of Turing Machine ("Church Turing Thesis")