Performance Evaluation of Distance-Hop Proportionality on Geometric Graph Models of Dense Sensor Networks

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Outline

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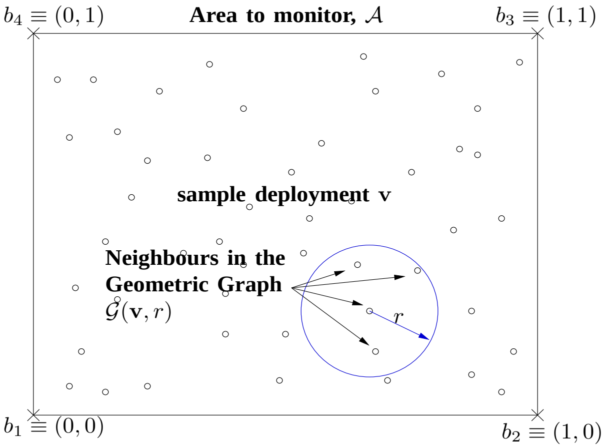
The Geometric Graph Model $\mathcal{G}(\mathbf{v}, r)$

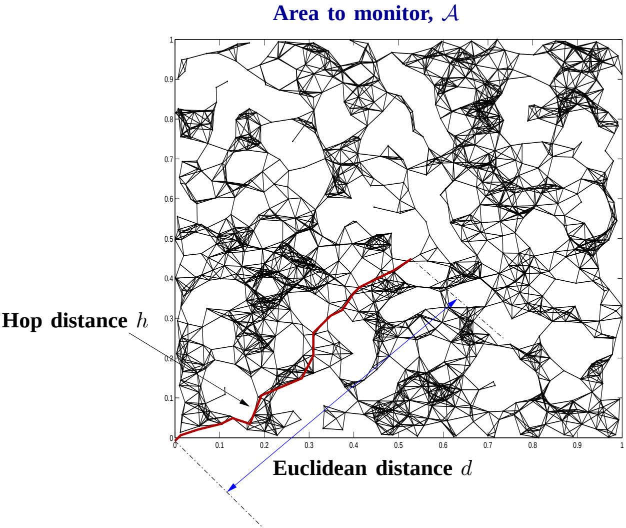
A Commonly used Model for Wireless Sensor Networks

- n nodes on a unit area, $\mathcal{A}$; locations: $\mathbf{v} = [v_1, v_2, \cdots, v_n] \in \mathcal{A}^n$



Node locations can be arbitrary or random

Hop Distance (HD) and Euclidean Distance (ED)



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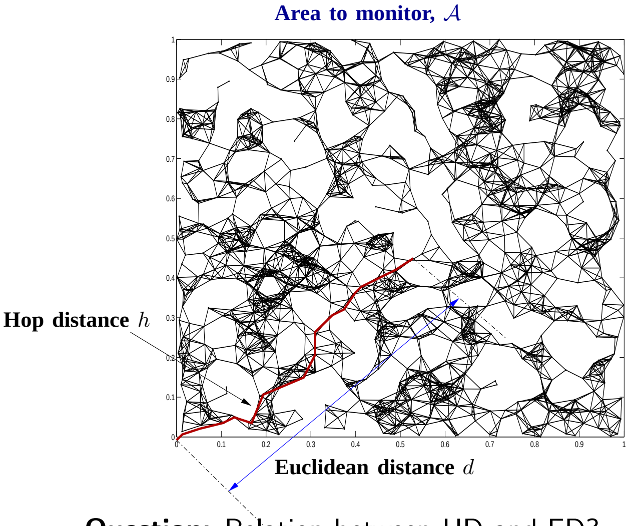


Question: Relation between HD and ED?

Motivation: GPS-free localisation

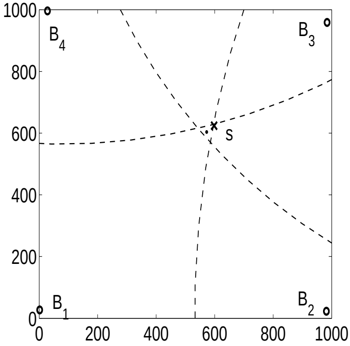
Outline

1. The Distance-Hop Proportionality Problem
   - Geometric Graphs and the HD-ED Problem
   - Motivation for the Problem: GPS-Free Localisation
   - Why Random Geometric Graphs
2. HD-ED Proportionality in a Random Geometric Graph (RGG)
3. Simulations Illustrating the Point-Node Theorem
4. Conclusion

A GPS-Free Localisation Algorithm: HCRL

- Hop Count Ratio-based Localisation (HCRL) [Yang et al. 2007]
- Assumption: $ED \propto HD$
- Hence,

$$\frac{d_{s,B_1}}{d_{s,B_2}} = \frac{h_{s.B1}}{h_{s,B_2}}$$



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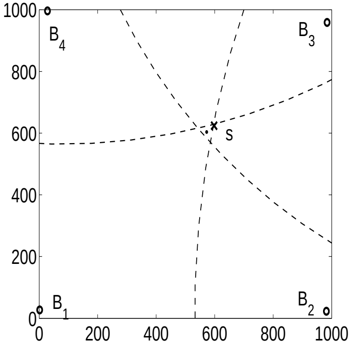
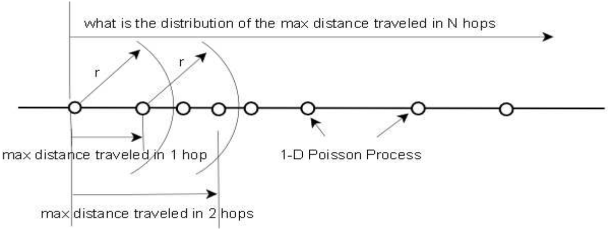


- Suppose that the location of s is (x, y)
- "Anchors" $B_k, 1 \leq k \leq 4$, at known locations (x_k, y_k)

$$\frac{\sqrt{(x - x_1)^2 + (y - y_1)^2}}{\sqrt{(x - x_2)^2 + (y - y_2)^2}} \approx \frac{h_{s.B1}}{h_{s,B_2}} \quad \Leftarrow \text{ Equation of a circle}$$

Literature: Distribution of the Distance Covered in k-Hops

- Vural and Ekici, Mobihoc 2005
  - Node locations: 1-dim Poisson process



  - Random Geometric Graph (RGG) on the line
  - Obtain an approximation to the distribution of the maximum distance traveled in a certain number of hops
- Dulman et al., 2006: Node locations: 2-dim Poisson process

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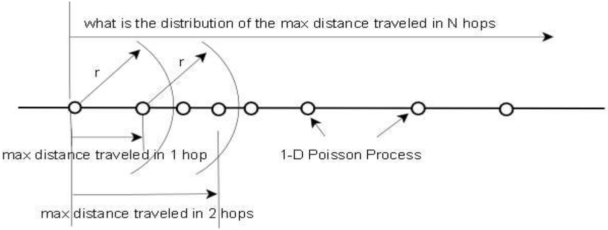


  - Random Geometric Graph (RGG) on the line
  - Obtain an approximation to the distribution of the maximum distance traveled in a certain number of hops
- Dulman et al., 2006: Node locations: 2-dim Poisson process
- We establish asymptotic proportionality of HD and ED, with a high probability

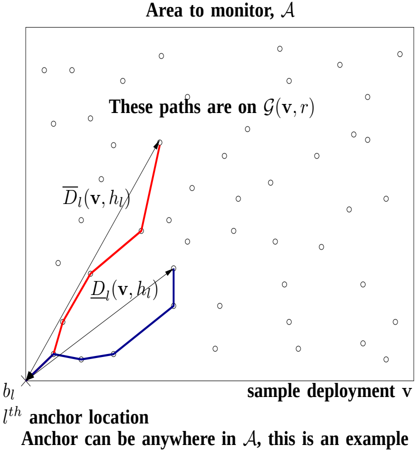
Outline

How Far is a Node that is h Hops from Anchor ℓ?

- $\mathcal{N} = \{1, 2, \cdots, n\}$, the set of the nodes
- $H_{\ell,i}(\mathbf{v})$ = hop distance of node i from anchor ℓ
- $D_{\ell,i}(\mathbf{v})$ = distance of node i from anchor ℓ

$$\overline{D}_\ell(\mathbf{v}, h_\ell) = \max_{\{i \in \mathcal{N} : H_{\ell,i}(\mathbf{v}) = h_\ell\}} D_{\ell,i}(\mathbf{v})$$

$$\underline{D}_\ell(\mathbf{v}, h_\ell) = \min_{\{i \in \mathcal{N} : H_{\ell,i}(\mathbf{v}) = h_\ell\}} D_{\ell,i}(\mathbf{v})$$



HD-ED Relationship in an Arbitrary Geometric Graph

Lemma

For arbitrary $\mathbf{v}$ *and* $h_\ell \geq 2$, $r < \underline{D}_\ell(\mathbf{v}, h_\ell) \leq \overline{D}_\ell(\mathbf{v}, h_\ell) \leq h_\ell r$ *and both bounds are sharp.*

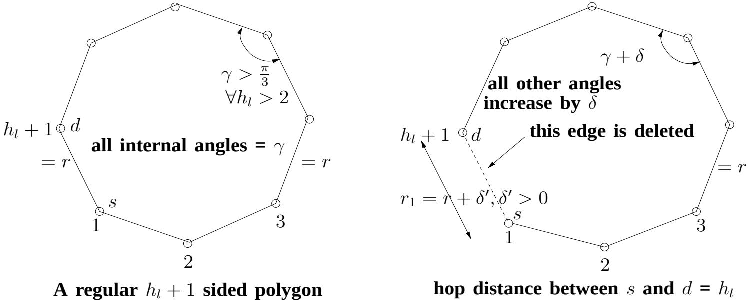


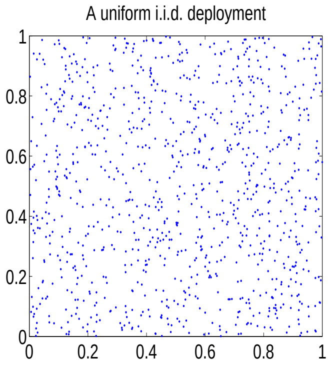
Figure: Node placement on the right achieves the lower bound of ED

- HD does not give useful information about ED in an arbitrary GG

Outline

The Random Geometric Graph (RGG) Setting

- n nodes; *Uniform i.i.d.* placement on unit area $\mathcal{A}$
- Random locations $\mathbf{V} = [V_1, V_2, \cdots, V_n] \in \mathcal{A}^n$
- $\mathbb{P}^n(.)$ is the probability measure
- The random geometric graph $\mathcal{G}(\mathbf{V}, r(n))$ is formed
  - $r(n) = c\sqrt{\frac{\ln n}{n}}$, $c > \frac{1}{\sqrt{\pi}}$
  - Ensures asymptotic connectivity with probability approaching 1



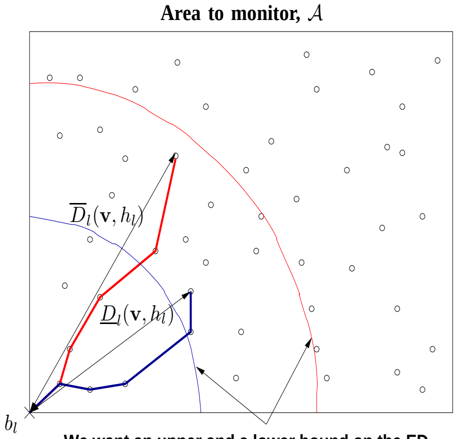
RGG: How Far Can a Node be that is h_ℓ Hops Away?

$$\overline{D}_\ell(\mathbf{v}, h_\ell) = \max_{\{i \in \mathcal{N} : H_{\ell,i}(\mathbf{v}) = h_\ell\}} D_{\ell,i}(\mathbf{v})$$

$$\underline{D}_\ell(\mathbf{v}, h_\ell) = \min_{\{i \in \mathcal{N} : H_{\ell,i}(\mathbf{v}) = h_\ell\}} D_{\ell,i}(\mathbf{v})$$

By the triangle inequality:

$$\overline{D}_\ell(\mathbf{v}, h_\ell) \leq h_\ell r(n)$$



- We want bounds on the Euclidean distance (ED) between a fixed point (say and anchor, b_ℓ) and all nodes at a hop-distance h_ℓ from the point

The "Point-Node" Theorem

$$E_{h_\ell}(n) := \{\mathbf{v} : (1-\epsilon)(h_\ell - 1)r(n) \le \underline{D}_\ell(\mathbf{v}, h_\ell) \le \overline{D}_l(\mathbf{v}, h_\ell) \le h_\ell r(n)\}$$

Theorem

For a given $1 > \epsilon > 0$, *and* $r(n) = c\sqrt{\frac{\ln n}{n}}$, $c > \frac{1}{\sqrt{\pi}}$,

$$\mathbb{P}^n(E_{h_\ell}(n)) = 1 - \mathcal{O}\left(\frac{1}{n^{g(\epsilon)c^2}}\right)$$

where

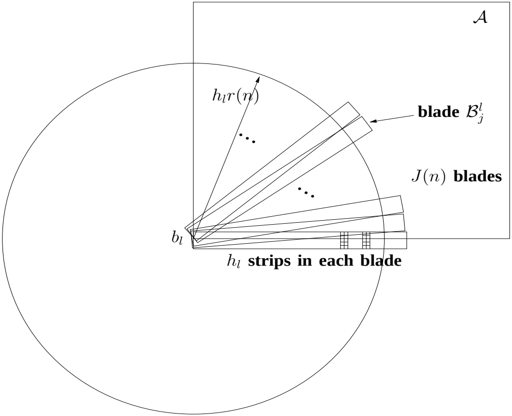
$$g(\epsilon) = q(\epsilon)\sqrt{1 - p^2(\epsilon)},$$

with $p(\epsilon) = \frac{1-\epsilon+\sqrt{(1-\epsilon)^2+8}}{4}$, $q(\epsilon) = \frac{-3(1-\epsilon)+\sqrt{(1-\epsilon)^2+8}}{4}$.

$$\textit{Hence,}\ \lim_{n\to\infty} \mathbb{P}^n(E_{h_\ell}(n)) = 1$$

Since $g(\epsilon) \downarrow$ as $\epsilon \downarrow$, the rate of convergence slows down as ϵ decreases.

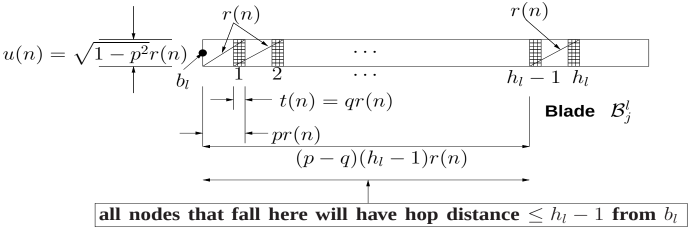
The "Point-Node" Theorem: Outline of the Proof (1/4)



- Circle of radius $h_\ell r(n)$ centered at the "point"
- Cover the circumference, within $\mathcal{A}$, by "blades," as shown
- Each blade is then covered with overlapping rectangles
- The overlaps are called "strips"

The "Point-Node" Theorem: Outline of the Proof (2/4)

- We take $0 < q < p < 1$; these will related to ϵ later



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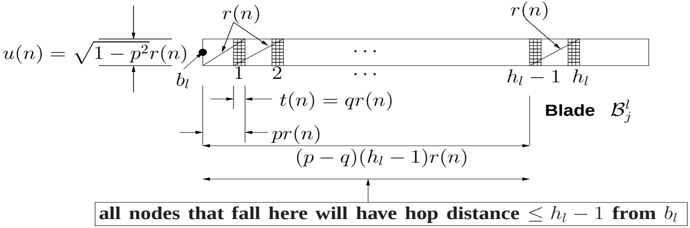


$$A_{i,j}^{\ell} = \{\mathbf{v} : \exists \text{ at least one node in the } i^{th} \text{ strip of } \mathcal{B}_j^{\ell}\}$$

$$\{\cap_{j=1}^{J(n)} \cap_{i=1}^{h_\ell - 1} A_{i,j}^{\ell}\}$$
$$\subseteq \quad \{\mathbf{v} : (p - q)(h_\ell - 1)r(n) \leq \underline{D}_\ell(\mathbf{v}, h_\ell) \leq \overline{D}_\ell(\mathbf{v}, h_\ell) \leq h_\ell r(n)\}$$

The "Point-Node" Theorem: Outline of the Proof (3/4)

$$
\begin{aligned}
\mathbb{P}^n\left(\cap_{j=1}^{J(n)}\cap_{i=1}^{h_\ell-1} A_{i,j}^\ell\right) &= 1-\mathbb{P}^n\left(\cup_{j=1}^{J(n)}\cup_{i=1}^{h_\ell-1} {A_{i,j}^\ell}^c\right)\\
&\geq 1-\sum_{j=1}^{J(n)}\sum_{i=1}^{h_\ell-1}\mathbb{P}^n\left({A_{i,j}^\ell}^c\right)\\
&\geq 1-(h_\ell-1)\left\lceil\frac{\pi h_\ell}{2\sqrt{1-p^2}}\right\rceil(1-u(n)t(n))^n\\
&\geq 1-(h_\ell-1)\left\lceil\frac{\pi h_\ell}{2\sqrt{1-p^2}}\right\rceil e^{-nu(n)t(n)}\\
&= 1-(h_\ell-1)\left\lceil\frac{\pi h_\ell}{2\sqrt{1-p^2}}\right\rceil e^{-nq\sqrt{1-p^2}r^2(n)}\\
&= 1-(h_\ell-1)\left\lceil\frac{\pi h_\ell}{2\sqrt{1-p^2}}\right\rceil n^{-q\sqrt{1-p^2}c^2}\overset{n\to\infty}{\longrightarrow}1
\end{aligned}
$$

The "Point-Node" Theorem: Outline of the Proof (4/4)

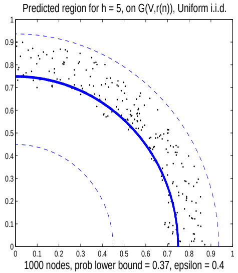
- We take $p - q = 1 - \epsilon$, and maximise $q\sqrt{1 - p^2}$
- Gives $p(\epsilon) = \frac{1-\epsilon+\sqrt{(1-\epsilon)^2+8}}{4}$, $q(\epsilon) = \frac{-3(1-\epsilon)+\sqrt{(1-\epsilon)^2+8}}{4}$
- Define $g(\epsilon) = q(\epsilon)\sqrt{1 - p^2(\epsilon)}$
- Hence,

$$\mathbb{P}^n\{\mathbf{v} : (1-\epsilon)(h_\ell - 1)r(n) \leq \underline{D}_\ell(\mathbf{v}, h_\ell) \leq \overline{D}_\ell(\mathbf{v}, h_\ell) \leq h_\ell r(n)\}$$

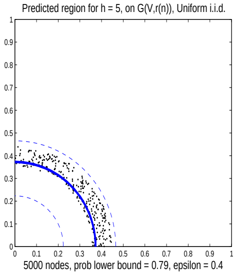
$$= 1 - \mathcal{O}\left(\frac{1}{n^{g(\epsilon)c^2}}\right)$$

Outline

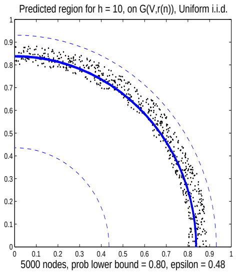
Simulation: $n = 1000, 5000, 5000$; $h_\ell = 5, 5, 10$ Hops



$\mathbb{P}^n(E_1(n)) \geq 0.37$



$\mathbb{P}^n(E_1(n)) \geq 0.79$



$\mathbb{P}^n(E_1(n)) \geq 0.80$

- $\epsilon = 0.4,\ r(n) = \frac{4}{\sqrt{\pi}}\sqrt{\frac{\ln n}{n}}$
- The dashed curves show the ED bounds given by the Point-Node Theorem
  - The probability lower bound from the theorem is shown
- The solid line shows the ED $(h_1 - 1)r(n)$

Observations and Extensions

- Observations from simulations
  - The bounds are valid, but
  - The lower bound $(1 - \epsilon)(h_\ell - 1)r(n)$ is quite loose, and
  - The bounds $[(h_\ell - 1)r(n), h_\ell r(n)]$ might be a good approximation

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  - The bounds $[(h_\ell - 1)r(n), h_\ell r(n)]$ might be a good approximation
- Extensions in the paper
  - RGG with a fixed radius r: Exponential convergence of the probability
  - RGG with *Randomized Lattice* deployment of nodes
    - ⋆ A similar point-node theorem is obtained
- Other extensions that we have shown
  - Node-node theorem, point-point theorem

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Conclusion, Applications, Future Work

Summary

- Assumed a Geometric Graph model of a Wireless Sensor Network
- HD is not a good measure of ED for *arbitrary* node placement
- Established high probability bounds on the ED, given the HD (h) between a fixed point and a node

$$(1-\epsilon)(h-1)r < \text{ED} \leq hr \text{ with high probability}$$

- Illustrated the theory with simulations

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Future Work

- Obtaining sharper bounds, perhaps by a different geometrical construction
- Improving the convergence rate result

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