Affine Maximizers in Domains with Selfish Valuations*

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Abstract

We consider the domain of *selfish* and *continuous* preferences over a "rich" allocation space and show that onto, strategyproof and *allocation non-bossy* social choice functions are affine maximizers. Roberts (1979) proves this result for a finite set of alternatives and an unrestricted valuation space. In this paper, we show that in a sub-domain of the unrestricted valuations with the additional assumption of *allocation non-bossiness*, using the richness of the allocations, the strategyproof social choice functions can be shown to be affine maximizers. We provide an example to show that allocation non-bossiness is indeed critical for this result. This work shows that an affine maximizer result needs certain amount of richness split across valuations and allocations.

1 Introduction

Allocating divisible resources or tasks among competing agents is a classical economic problem. The proliferation of the Internet and the rapid development of computing power have created a marketplace for electronic resources. The consumers in these markets are typically individuals, small or medium businesses who pay the service providers for Internet connectivity. The frequencies of mobile telephony or 2G/3G bandwidth are scarce resources and the businesses and service providers often have different valuations for the resources which are private information. Designing mechanisms that reveal the agents' demands truthfully is therefore an important problem. There is an extensive literature that deals with such applications (Yaïche et al., 2000; Sahasrabudhe and Kar, 2008; Wang et al., 2012). A similar resource allocation problem arises in the context of *cloud computing* and *cloud storage*.

All these markets use money extensively for their services, and agents privately observe their valuations or the costs of their tasks. Following the vast literature in this area, we use quasi-linear utilities to model the payoffs. In addition, the valuations are private and selfish (an agent's valuation is a function of her own resource share). However, the valuations are not assumed to be increasing in the amount of the resources consumed. The following example shows that this situation occurs often in practice. Consider an agent who requires a certain amount of resource

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(to run her business optimally) say $\lambda \in [0, 1]$ fraction of the divisible resource. However, there is a resource maintenance cost that increases proportionally with the allocated resource. As long as her allocation is $\delta(< \lambda)$, her returns are more than the cost and the valuation increases. But if she is allocated more than a share of λ, the cost of maintaining the resource is more than the returns and her valuation decreases. So the valuation has a peak in the amount of the resource share. In a similar way we can construct examples where multiples of a particular amount of resource share is desired by an agent and therefore the valuation can have multiple peaks. The main question we ask in this paper is:

> What is the class of strategyproof social choice functions when agents have arbitrary selfish valuations over divisible commodities?

The answer is provided as the main result of this paper that shows that the *strategyproof* and *allocation non-bossy* mechanisms in this model must belong to the well-known *affine maximizer* class. An immediate consequence of this result is that the payments are of a specific functional form.

In the following two subsections, we present the related literature and a panoramic sketch of the proof respectively.

1.1 Relationship to the literature

Our paper is related to two different strands in the literature. The first deals with results that characterize social choice functions (SCFs) that are *affine maximizers*. [1] The original result appears in Roberts (1979). It considers a finite set of allocations A (with $|A| \geq 3$) and an unrestricted valuation domain. According to the result, an onto and strategyproof SCF must be an affine maximizer. The case with $|A| = 2$ differs significantly from the $|A| \geq 3$ case and is addressed in Marchant and Mishra (2014). They show that the class of onto, strategyproof SCFs expands to the class of *generalized utility function maximizers*. If the SCFs are required to satisfy an additional independence condition, this class reduces to the class of affine maximizers. Carbajal et al. (2013) extend this result to an infinite allocation space. They consider a compact set of allocations and unrestricted continuous valuations and show that the affine maximizer result continues to hold. All the results mentioned above require "sufficient" richness in the valuation space. In particular, the valuation of an agent is required to depend on the allocations of *all* agents and therefore not compatible with *selfish* valuations.

The assumption of selfish valuations constitutes a domain restriction; consequently the class of strategyproof SCFs expands beyond that of the class of affine maximizers. This has been shown in several papers. Lavi et al. (2003) considers combinatorial auction domains. They show that strategyproof SCFs are 'almost affine maximizers' provided they satisfy an *independence of irrelevant alternatives* assumption. Mishra and Quadir (2014) characterizes strategyproof SCFs for single object auctions. They show that this is a larger class than that of affine maximizers even with the non-bossiness assumption. There are other characterization results for restricted domains, e.g., Maya and Nisan (2012) for the two player cake-cutting problem, that show that the class of strategyproof SCFs is larger than that of the affine maximizer class.

Our paper is also related to the literature on the division model introduced in Sprumont (1991). In the single dimensional version of this model, a task has to be allocated amongst several agents.

[1]The term 'affine maximizer' was coined in Meyer-ter Vehn and Moldovanu (2002).

Monetary compensation is not allowed and valuations are assumed to be single-peaked. A salient allocation rule in this setting is *uniform* rule characterized in Sprumont (1991) and in several other papers using a variety of axioms (see, e.g., Ching (1994)). Recently, this model has been extended to multiple dimensions (Cho and Thomson, 2013; Morimoto et al., 2013).

Our paper differs from both strands of the literature with respect to the model and the results. We consider a task / object allocation space that is *separable* into allocations for each agent. On this domain, we consider *arbitrary selfish valuations*. The valuation space is a strict subclass of the unrestricted valuations, but is different from the sub-domains considered in the literature above. Furthermore, in contrast to the affine maximizer literature, we consider a 'rich' allocation space - the set of allocations is a compact subset of $[0,1]^{n\times m}$. In contrast to the division model, single-peakedness is replaced by monetary transfers in quasi-linear form.

In this model, we show that an onto, strategyproof SCF belongs to the class of affine maximizers if they satisfy the additional assumption of *allocation non-bossiness*. We observe that if 'richness' of the valuation domain is complemented with the 'richness' in the allocation space, then the affine maximizer result is restored. We believe that this observation is important and novel.

1.2 Brief overview of the proof

The affine maximizer theorem in Roberts (1979) has been revisited several times with different proofs (Dobzinski and Nisan, 2009; Lavi et al., 2009; Mishra and Sen, 2012; Carbajal et al., 2013). Our proof builds on the arguments in Roberts (1979), Lavi et al. (2009) and Carbajal et al. (2013). The first two papers consider finite alternative space, while the last considers an uncountable compact alternative space with continuous valuations. As we have remarked earlier, all these papers require richness in the valuation space that is ruled out by the selfish valuations assumption. A critical element of these proofs is the construction of a *value difference set* for any two distinct allocations which is then shown to be a half-space. In the selfish valuation domain, this idea cannot be applied directly since there can exist two distinct allocations x and y whose i-th components x_i and y_i are identical. The value difference set for these two allocations will have zero at the i-th component for all vectors and will cease to have the half-space property.

In order to overcome this difficulty, we construct a subset of the allocation space called the *distinct component set (DCS)* that has distinct elements in each of the components. We show the existence of a maximal DCS and show that it is *dense* [2] in the allocation space. Our proof proceeds by establishing the affine maximizer result on this *maximal and dense* DCS and then extending it to the full allocation space. The allocation non-bossiness assumption is used critically to establish the appropriate monotonicity condition required for the result.

The rest of the paper is organized as follows. We present the formal model in Section 2, state the main result in Section 3 and provide its proof in Section 4. We make a remark about the revenue equivalence result in Section 5. In Section 6, we discuss a more general result that can be proved using similar arguments. We conclude the paper in Section 7.

2 The Model and Definitions

Let $N = \{1, \ldots, n\}, n \geq 2$ be the set of agents. There is a set $J = \{1, \ldots, m\}$ of perfectly divisible tasks (or objects) to be allocated among the agents. Let $x_{ij} \in [0,1], i \in N, j \in J$ be the fraction of

[2] A set D is dense in a metric space X, if every $x \in X$ either belongs to D or is a limit point of D.

task j assigned to agent i. We assume that each task is fully allocated, i.e., $\sum_{i \in N} x_{ij} = 1, \forall j \in J$. Let $x_i \equiv (x_{i1}, \ldots, x_{im}) \in [0,1]^m$ denote the fractions of the tasks allocated to agent i. An allocation x is therefore a matrix where the rows and columns correspond to the agents and tasks respectively. The (i,j)-th element of this matrix is x_{ij}, as explained above. Formally, we define the set of allocations to be:

$$A = \{z \in [0,1]^{n \times m} : \sum_{i \in N} z_{ij} = 1, \forall j \in J\}$$

Observe that the allocations in A are *separable*, i.e., it can be split into allocations for each agent, and is *compact*.

A *valuation function* for agent i is a *continuous* [3] map $u_i : A \to \mathbb{R}$. It is *selfish* if, $u_i(x_i, x_{-i}) = u_i(x_i, x'_{-i})$, for all $(x_i, x_{-i}), (x_i, x'_{-i}) \in A$. Let $x \in A$. If u_i is selfish, we can write $u_i(x) = u_i(x_i)$ without loss of generality. Selfishness implies that the valuation of an agent depends only on the task portions assigned to her, and does not depend on the task portions assigned to other agents.

Definition **1 (Arbitrary Selfish Valuations)** *The set of* all *selfish valuations is the domain of* arbitrary selfish valuations *(ASV). It is denoted by U.*

We *do not* put any further restrictions on valuation functions. In particular, we do not assume that the valuation function is *increasing* (i.e., $u_i(x_i) \geq u_i(x'_i)$ whenever $x_i > x'_i$ [4]) or *additively separable across tasks* (i.e., for every agent i there exists $u_{ij} : [0,1] \to \mathbb{R}$ for every task j such that $u_i(x_i) = \sum_{j \in J} u_{ij}(x_{ij})$). Note that imposing these conditions would restrict the domain beyond ASV. We are not sure whether the affine maximizer result continues to hold in these further restricted domains.

In addition to the utility that an agent derives from the tasks assigned to her, she can also be compensated with (or charged) money. Moreover, the overall utility function is quasi-linear in money, i.e., given by $u_i(x_i) + p_i$, where p_i is agent i's monetary compensation.

We assume that the agent valuations are private information and must be elicited. The goal of the paper is to identify collective outcomes that induce agents to reveal their private information truthfully. The following definitions are standard in mechanism design literature.

Definition **2 (Social Choice Function)** *A* social choice function *(SCF) F is a mapping from the set of valuation profiles to the set of allocations, i.e., $F : U^n \to A$.*

Definition **3 (Strategyproofness)** *A SCF F is* strategyproof *if there exist transfers $p_i : U^n \to \mathbb{R}, i \in N$, such that for all $u = (u_i, u_{-i}) \in U^n$,*

$$u_i(F(u_i, u_{-i})) + p_i(u_i, u_{-i}) \geq u_i(F(u'_i, u_{-i})) + p_i(u'_i, u_{-i}), \quad \forall\, u'_i \in U, i \in N.$$

A strategyproof SCF induces truth-telling in dominant strategies.

The problem of characterizing strategyproof SCFs in allocation problems with selfish preferences presents special difficulties because agents can affect the allocations of other agents without affecting their own. An axiom that addresses this issue in a "minimal" way is that of *allocation non-bossiness.* The non-bossiness axiom was introduced in Satterthwaite and Sonnenschein (1981), and has been used widely in the literature on incentive compatibility. [5]

[3] A function $f : A \subseteq \mathbb{R}^m \to \mathbb{R}$ is continuous at a point $x_0 \in A$ if $\forall \epsilon > 0$ there exists $\delta_{x_0} > 0$ such that whenever $||x - x_0|| < \delta_{x_0}$, $||f(x) - f(y)|| < \epsilon$.

[4] Here $x_i > x'_i \Rightarrow x_{ij} > x'_{ij}$, for all $j \in J$.

[5] For matching, see e.g., (Svensson, 1999; Pápai, 2000), and for exchange economies, see e.g., (Goswami et al., 2014; Momi, 2013). For a comprehensive discussion of the non-bossiness axiom, see Thomson (2014).

Definition 4 **(Allocation Non-Bossiness)** *An SCF F satisfies* allocation non-bossiness (ANB) *if $\forall i \in N$, $\forall u_{-i} \in U^{n-1}$ and $\forall u_i, u_i' \in U$ with $F_i(u_i, u_{-i}) = F_i(u_i', u_{-i})$ implies $F(u_i, u_{-i}) = F(u_i', u_{-i})$, where $F_i(\cdot)$ is the i-th component of $F(\cdot)$.*

Suppose an agent's allocation is unaffected by a change in her valuation. According to ANB, this must not change the allocation of any other agent. A well-known fact about strategyproofness is that a change in an agent's valuation that does not affect her allocation, does not also affect her payment. Therefore, the "if" part of non-bossiness together with strategyproofness ensures that the change in the agent's valuation does not change her overall utility. In such a situation, the *allocations* of other agents is not allowed to change. We place no restrictions on the payments of the other agents, so that their utilities may change.

Remarks:

1. It is important to distinguish our version of non-bossiness from a related notion which we refer to as *outcome non-bossiness*. In a quasi-linear domain, an outcome comprises of the allocation and payments. An SCF is outcome non-bossy if the following holds: if an agent does not change her outcome by changing her own valuation, she does not change the outcomes of the other agents.

2. Outcome non-bossiness and strategyproofness imply allocation non-bossiness. Outcome non-bossiness is therefore a stronger requirement than allocation non-bossiness. In fact, affine maximizers (defined below), in general, do not satisfy outcome non-bossiness.

3. Our ANB condition is *weaker* than the *independence of irrelevant alternatives* (IIA) which was used in Lavi et al. (2003) for the characterization of *almost affine maximizers* in combinatorial auctions. The main difference lies in the 'if' part of the definitions. The ANB property requires the following: if the *allocation* of an agent does not change due to change in her valuation, then the allocation of no other agent should change. On the other hand, IIA requires the following: if the difference in the *valuations* due to two different social choice outcomes remain same for all agents, then the outcomes must be the same. Formally, $\forall i \in N$, if $u_i(F(u)) - u_i(F(u')) = u_i'(F(u)) - u_i'(F(u'))$, then $F(u) = F(u')$, for all $u, u' \in U^n$. Below, we show that IIA implies ANB but not the converse.

   IIA implies ANB: We prove that $\neg$ANB $\Rightarrow$ $\neg$IIA. Suppose there exists an agent i and a pair of valuations u_i, u_i' for the agent and a valuation profile for other agents u_{-i} such that $F_i(u_i, u_{-i}) = F_i(u_i', u_{-i})$, but $F(u_i, u_{-i}) \neq F(u_i', u_{-i})$, i.e., allocation is different of at least one agent $j \neq i$. Hence i is bossy and F is not ANB. Note that, for agent i,

$$\begin{aligned} u_i(F(u_i, u_{-i})) - u_i(F(u_i', u_{-i})) &= u_i(F_i(u_i, u_{-i})) - u_i(F_i(u_i', u_{-i})) \\ &= 0 \\ &= u_i'(F_i(u_i, u_{-i})) - u_i'(F_i(u_i', u_{-i})) \\ &= u_i'(F(u_i, u_{-i})) - u_i'(F(u_i', u_{-i})). \end{aligned}$$

   The first equality is due to the fact that the valuations are selfish. The second equality holds because the allocations are same for agent i (by assumption). The last two equalities are consequences of selfish valuations again. For all agents $j \neq i$, the equality is satisfied since their valuations are unchanged. Therefore, the 'if' part of IIA is satisfied for the valuation profiles (u_i, u_{-i}) and (u'_i, u_{-i}). However, $F(u_i, u_{-i}) \neq F(u'_i, u_{-i})$. Hence, F does not satisfy IIA.

ANB but not IIA: Fix two allocations (a_i, a_{-i}) and (b_i, a_{-i}), $b_i \neq a_i$. Define a specific valuation of agent i as follows:

$$\hat{u}_i(z_i) = \begin{cases} r_a & \text{when } z_i = a_i \\ r_b & \text{when } z_i = b_i \\ 0 & \text{otherwise.} \end{cases}$$

Construct a SCF F such that:

$$F(u_i, u_{-i}) = \begin{cases} (a_i, a_{-i}) & \forall u_i \neq \hat{u}_i, \forall u_{-i} \\ (b_i, a_{-i}) & u_i = \hat{u}_i, \forall u_{-i} \end{cases}$$

All the components of F except i are always set to a_{-i}, and the i-th component is a_i for all valuations of agent i except $\hat{u}_i$, at which it is b_i. Note that F is only affected by the valuation of agent i; when agent i changes her valuation from $\hat{u}_i$ to anything else or vice versa, her allocation also changes. Hence the ANB condition is satisfied.

Now consider two valuation profiles $(\hat{u}_i, u_{-i})$ and $(\tilde{u}_i, , u_{-i})$ where $\hat{u}_i$ is as defined before and $\tilde{u}_i$ is defined for some $\delta \neq 0$ as follows:

$$\tilde{u}_i(z_i) = \begin{cases} r_a + \delta & \text{when } z_i = a_i \\ r_b + \delta & \text{when } z_i = b_i \\ 0 & \text{otherwise.} \end{cases}$$

We see that the 'if' part of the IIA condition is satisfied: for agent i,

$$\hat{u}_i(F(\hat{u}_i, u_{-i})) - \hat{u}_i(F(\tilde{u}_i, u_{-i})) = r_b - r_a = \tilde{u}_i(F(\hat{u}_i, u_{-i})) - \tilde{u}_i(F(\tilde{u}_i, u_{-i})),$$

and for all agents $j \neq i$,

$$u_j(F(\hat{u}_i, u_{-i})) - u_j(F(\tilde{u}_i, u_{-i})) = 0 = u_j(F(\hat{u}_i, u_{-i})) - u_j(F(\tilde{u}_i, u_{-i})).$$

However, $F(\hat{u}_i, u_{-i}) \neq F(\tilde{u}_i, u_{-i})$. Hence F does not satisfy IIA.

A salient class of SCFs introduced in Roberts (1979) is that of *affine maximizers*, defined as follows.

Definition **5 (Affine Maximizers)** *An SCF $F : U^n \to A$ belongs to the class of* affine maximizers *if there exists $w_i \geq 0, i \in N$, not all zero, and a continuous function $\kappa : A \to \mathbb{R}$ such that,*

$$F(u) \in \underset{x \in A}{\operatorname{argmax}} \left(\sum_{i \in N} w_i u_i(x) + \kappa(x) \right).$$

The continuity of the valuations and the compactness of the set of allocations ensures that the class of affine maximizers is non-empty.

3 Main Result

In this section, we present the central result of the paper.

Theorem **1 (Affine Maximizers for Selfish Valuations)** *If $n \geq 3$, every* onto [6], ANB *and* strategyproof *SCF $F : U^n \to A$ is an affine maximizer. If $n = 2$, the result holds without the ANB assumption.*

The ASV assumption ensures that onto affine maximizers exist. Our next claim shows the existence of affine maximizers with the properties mentioned in Theorem 1. The proof is deferred to the Appendix.

Claim **1** *There exist affine maximizers that are onto, strategyproof and ANB.*

The ANB assumption is necessary for our result. We provide an example to show the existence of non-affine maximizer SCFs that are strategyproof and bossy.

Example **1 (Serial Dictator with a Threshold)** *For simplicity, we illustrate a SCF F with three agents, $N = \{1, 2, 3\}$ and one task, i.e., $m = 1$ (see Figure 1). The arguments can clearly be extended to a setting with arbitrary n and m. Let t be an arbitrary real number referred to as the threshold. The SCF F is defined below.*

For any valuations profile (u_1, u_2, u_3), the SCF picks $x_1^* \in \mathrm{argmax}_{x_1 \in [0,1]} u_1(x_1)$. Ties are broken in favor of the largest value of x_1^*. In order to determine the allocations of agents 2 and 3, we need to consider the two following cases.

Case 1: $u_1(x_1^*) \leq t$: *Then $x_2^* \in \mathrm{argmax}_{x_2 \in [0,1-x_1^*]} u_2(x_2)$ with ties broken as before, and $x_3^* = 1 - x_1^* - x_2^*$.*

Case 2: $u_1(x_1^*) > t$: *In this case, the role of agents 2 and 3 is reversed, i.e., $x_3^* \in \mathrm{argmax}_{x_3 \in [0,1-x_1^*]} u_3(x_3)$ with ties broken as before, and $x_2^* = 1 - x_1^* - x_3^*$.*

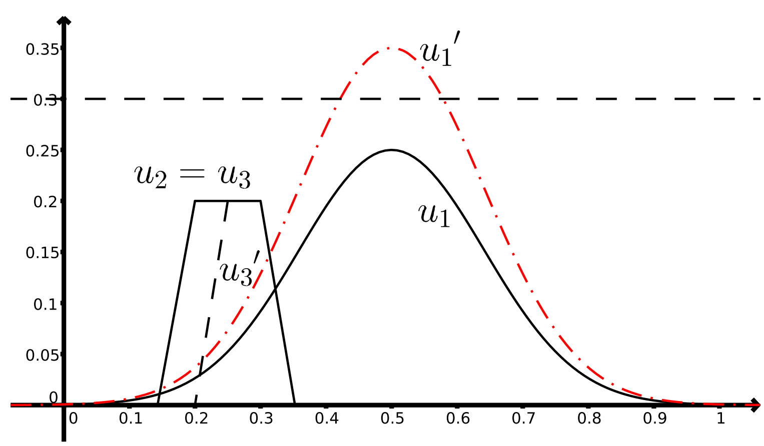

Figure 1: An example to illustrate the necessity of ANB.

Clearly F is onto. It is bossy because agent 1 can change the allocations of agents 2 and 3 without changing her own allocation. This is illustrated in Figure 1 - for the valuation profiles (u_1, u_2, u_3) and (u_1', u_2, u_3), the allocations are $[0.5\ 0.3\ 0.2]^\top$ and $[0.5\ 0.2\ 0.3]^\top$ respectively.

[6]An SCF $F : U^n \to A$ is *onto*, if for all $x \in A$, there exists $u \in U^n$ such that $F(u) = x$.

Strategyproofness of F follows from standard arguments. We claim that F is not an affine maximizer. To see this, we repeat the arguments from Equations 9 to 10 with $x = [0.5\ 0.3\ 0.2]^\top$ and $y = [0.5\ 0.2\ 0.3]^\top$ to get,

$$\begin{aligned} & w_2 u_2(0.3) + w_3 u_3(0.2) + \kappa([0.5\ 0.3\ 0.2]^\top) = w_2 u_2(0.2) + w_3 u_3(0.3) + \kappa([0.5\ 0.2\ 0.3]^\top) \\ \Rightarrow\ & \kappa([0.5\ 0.3\ 0.2]^\top) = \kappa([0.5\ 0.2\ 0.3]^\top) \quad \text{since } u_2(0.3) = u_2(0.2), u_3(0.2) = u_3(0.3). \end{aligned}$$

Pick u'_3 which agrees with u_3 everywhere except $[0.15, 0.25]$ and is shown by the dashed line in the figure. By definition, $F(u_1, u_2, u'_3) = [0.5\ 0.3\ 0.2]^\top$. However,

$$\begin{aligned} & w_1 u_1(0.5) + w_2 u_2(0.3) + w_3 u'_3(0.2) + \kappa([0.5\ 0.3\ 0.2]^\top) \\ & < w_1 u_1(0.5) + w_2 u_2(0.2) + w_3 u'_3(0.3) + \kappa([0.5\ 0.2\ 0.3]^\top), \end{aligned}$$

since $u_2(0.2) = u_2(0.3) = 0.2$ but $0.2 = u'_3(0.3) > u'_3(0.2) = 0$ and κ is same at these two allocations. Hence, F is not an affine maximizer. □

4 Proof of the Theorem

In order to present the proof, we need to define an important property. Since allocations are decomposable into components for every agent and valuations are selfish, it is possible for the component of agent i to be identical in two different allocations. To account for this possibility we need the following modification of the notion of positive association of differences (PAD) which appears in Roberts (1979).

Definition 6 **(PAD-DC)** *An SCF F satisfies* positive association of differences for distinct components (PAD-DC) *if $\forall u, u' \in U^n$,*

$$[F(u) = x \text{ and } u'_i(x_i) - u_i(x_i) > u'_i(y_i) - u_i(y_i), \forall y \in A \setminus \{x\} \text{ s.t. } y_i \neq x_i, \forall i \in N] \Rightarrow [F(u') = x].$$

Let x be an allocation and u be a valuation profile such that $F(u) = x$. Such a profile exists because of the onto-ness of F. If we consider all other allocations in A, there will be an allocation $z \in A$ such that $x_j = z_j$ for some $j \in N$. Since valuations are selfish, $u'_j(x_j) - u_j(x_j) = u'_j(z_j) - u_j(z_j)$ and the conclusion $F(u') = x$ cannot be inferred from PAD. However, if $u'_i(x_i) - u_i(x_i) > u'_i(y_i) - u_i(y_i), \forall y_i \neq x_i$, we *can* conclude $F(u') = x$ from PAD-DC. Of course, for the ASV domain, PAD-DC implies PAD, i.e., PAD-DC is a stronger property than PAD.

The next result shows that PAD-DC is an implication of ANB and strategyproofness.

Lemma **1** *If $n \geq 3$, every* ANB, strategyproof *SCF satisfies PAD-DC. For $n = 2$, every* strategyproof *SCF satisfies PAD-DC.*

Proof: We prove the lemma in three steps. Let F be strategyproof and ANB.

Step 1: Pick any $i \in N$ and let $(u_i, u_{-i}), (u'_i, u_{-i}) \in U^n$ be such that $F(u) = x$, and suppose $\forall y \in A \setminus \{x\}$, $u'_i(x_i) - u_i(x_i) > u'_i(y_i) - u_i(y_i)$, for all $y_i \neq x_i$. We claim $F_i(u'_i, u_{-i}) = x_i$.

Suppose not, i.e., $F_i(u'_i, u_{-i}) = z_i \neq x_i$. By assumption, $u'_i(x_i) - u_i(x_i) > u'_i(z_i) - u_i(z_i)$. However, strategyproofness of F implies $u'_i(x_i) - u_i(x_i) \leq u'_i(z_i) - u_i(z_i)$ [7], which is a contradiction. Hence $F_i(u'_i, u_{-i}) = x_i$.

Step 2: Consider the case $n \geq 3$. Since Step 1 is true for every $i \in N, u_i, u'_i \in U, u_{-i} \in U^{n-1}$, ANB implies that $F(u'_i, u_{-i}) = x$. If $n = 2$, it is trivially true since there are only two components in each allocation x_1 and x_2, which are row vectors of length m and their sums equal the 'all-one' vector of length m. Hence, if x_1 does not change, nor does x_2.

Step 3: We use these arguments for every player. Fix $u, u' \in U^n$ as in the definition of PAD-DC, i.e., $F(u) = x$ and $u'_i(x_i) - u_i(x_i) > u'_i(y_i) - u_i(y_i)$, $\forall y \in A \setminus \{x\}, y_i \neq x_i, \forall i \in N$. We have shown that $F(u_i, u_{-i}) = x$ implies $F(u'_i, u_{-i}) = x$. We repeat Steps 1 and 2 above for the transition from the value profile (u'_i, u_{-i}) to (u'_i, u'_j, u_{-ij}) and conclude that $F(u'_i, u'_j, u_{-ij}) = x$. Proceeding in this manner we conclude that $F(u') = x$, as needed. ■

The next claim will be used repeatedly in the proof. It is a counterpart of Claim 1 in Lavi et al. (2009). Their argument cannot be used directly because of the domain restriction used in this paper.

Claim **2** *Let F satisfy PAD-DC, and fix $u, u' \in U^n$. Then,*

$$[F(u) = x \text{ and } u'_i(x_i) - u'_i(y_i) > u_i(x_i) - u_i(y_i) \text{ for some } y \in A \setminus \{x\} \text{ s.t. } y_i \neq x_i, \forall i \in N] \Rightarrow [F(u') \neq y]$$

Proof: Suppose not, i.e., $F(u') = y$. We will construct a valuation $u''_i \in U, i \in N$ such that an application of PAD-DC to the transition $u' \to u''$ will yield $F(u'') = y$. On the other hand, its application to the transition $u \to u''$ would imply $F(u'') = x$, leading to a contradiction.

In order to do that, we need to construct u'' to satisfy:

$$u''_i(y_i) - u'_i(y_i) > u''_i(z_i) - u'_i(z_i), \forall z_i \neq y_i, \forall i \in N \tag{1}$$

and,

$$u''_i(x_i) - u_i(x_i) > u''_i(z_i) - u_i(z_i), \forall z_i \neq x_i, \forall i \in N. \tag{2}$$

Clearly, if Equation (1) holds, then applying PAD-DC we get $F(u'') = y$, and similarly, if Equation (2) holds, then we get $F(u'') = x$.

Pick a pair of allocations x and y such that the LHS of the implication of the claim occurs, i.e., $F(u) = x$ and $u'_i(x_i) - u'_i(y_i) > u_i(x_i) - u_i(y_i)$ whenever $y_i \neq x_i, \forall i \in N$. A simple rearrangement of the terms yields, $\beta = u_i(x_i) - u'_i(x_i) < u_i(y_i) - u'_i(y_i) = \alpha$. Let $\delta = \alpha - \beta$ which is positive by assumption. The difference of this u''_i from that in Lavi et al. (2009) is that the construction here must also satisfy continuity of u''_i. Choose u''_i as follows:

[7] Let $p_i(\cdot), i \in N$ be the payment function that makes F strategyproof. The required inequality is obtained by summing the following inequalities which are consequences of strategyproofness:

$$u_i(x_i) + p_i(u_i, u_{-i}) \geq u_i(z_i) + p_i(u'_i, u_{-i}),$$
$$u'_i(z_i) + p_i(u'_i, u_{-i}) \geq u'_i(x_i) + p_i(u_i, u_{-i}).$$

$$u_i''(z_i) = \begin{cases} u_i'(x_i) + \beta + \delta/4 & \text{for } z_i = x_i \\ u_i'(y_i) + \alpha - \delta/4 & \text{for } z_i = y_i \\ \min\{u_i''(y_i) - (u_i'(y_i) - u_i'(z_i)) - \epsilon||y_i - z_i||, & \\ \qquad u_i''(x_i) - (u_i(x_i) - u_i(z_i)) - \epsilon||x_i - z_i||\} & \text{where } 0 < \epsilon < \frac{\delta}{2\sqrt{m}} \text{ and } z_i \neq x_i, y_i \end{cases} \tag{3}$$

We claim that the function u_i'' is continuous. Consider the third case in the above definition of the function u_i''. At the points $z_i \neq x_i, y_i$, the function is continuous since $u_i''(y_i)$ and $u_i''(x_i)$ are constants and all other functions in the two expressions of the min are continuous, and the minimum of two continuous functions is continuous. Therefore, we need to show that u_i'' is continuous at x_i and y_i as well.

To do this, let us take the limits of the two expressions in the min function when $z_i \neq x_i, y_i$ for $z_i \to y_i$ (the case for $z_i \to x_i$ is similar and is omitted here). Denote the two expressions in the min function as follows:

$$\begin{aligned} f_1(z_i) &= u_i''(y_i) - (u_i'(y_i) - u_i'(z_i)) - \epsilon||y_i - z_i|| \\ f_2(z_i) &= u_i''(x_i) - (u_i(x_i) - u_i(z_i)) - \epsilon||x_i - z_i|| \end{aligned}$$

For continuity, the limiting value of u_i'' for $z_i \to y_i$ should be $u_i''(y_i)$, i.e., $f_1(y_i)$ should be the min of $\{f_1(y_i), f_2(y_i)\}$. Taking limit $z_i \to y_i$, the expression $f_2(z_i) - f_1(z_i)$ becomes:

$$\begin{aligned} & u_i''(x_i) - (u_i(x_i) - u_i(y_i)) - \epsilon||x_i - y_i|| - u_i''(y_i) \\ &= u_i''(x_i) - u_i''(y_i) - (u_i(x_i) - u_i(y_i)) - \epsilon||x_i - y_i|| \\ &= u_i'(x_i) - u_i'(y_i) + \beta - \alpha + \delta/2 - (u_i(x_i) - u_i(y_i)) - \epsilon||x_i - y_i|| \\ &= (u_i'(x_i) - u_i(x_i)) + u_i(y_i) - u_i'(y_i) + \beta - \alpha + \delta/2 - \epsilon||x_i - y_i|| \\ &= -\beta + \alpha + \beta - \alpha + \delta/2 - \epsilon||x_i - y_i|| \\ &= \delta/2 - \epsilon||x_i - y_i|| \\ &> 0 \text{ since } ||x_i - y_i|| \leq \sqrt{m} \text{ and } \epsilon < \frac{\delta}{2\sqrt{m}} \end{aligned}$$

So the min of f_1 and f_2 for $z_i \to y_i$ is f_1, and $\lim_{z_i \to y_i} f_1(z_i) = u_i''(y_i)$. Hence $\lim_{z_i \to y_i} u_i''(z_i) = u_i''(y_i)$. Similarly, we can show, $\lim_{z_i \to x_i} u_i''(z_i) = u_i''(x_i)$. Hence u_i'' is continuous.

By construction, it is clear that u_i'' satisfies Equations (1) and (2) for all $z_i \neq x_i, y_i$. This can be seen by noticing the third condition of the definition of u_i''. We need to additionally show that Equation (1) is satisfied at $z_i = x_i$ and Equation (2) is satisfied at $z_i = y_i$. We show that Equation (1) is satisfied at $z_i = x_i$. The other case is similar and hence omitted. The LHS of Equation (1) is $\alpha - \delta/4$ by definition of u_i'', while the RHS at $z_i = x_i$ is $u_i''(x_i) - u_i'(x_i) = \beta + \delta/4$. Clearly, $\alpha - \delta/4 > \beta + \delta/4$ and hence Equation (1) is satisfied at $z_i = x_i$. Hence, we have proved the claim. ■

One of the central ideas of our proof is to show that F is an affine maximizer on a suitably chosen dense subset of the allocation set A. The proof is completed by extending the result to the entire set of allocations using the continuity of the valuations. The following definition is critical for the choice of the dense subset.

Definition **7 (Distinct Component Set)** *A set $S \subseteq A$ is called a* distinct component set (DCS) *if for every distinct $x, y \in S$, $x_i \neq y_i$, $\forall i \in N$.*

We consider the DCSs that contain a specific allocation x. Denote the set of all DCSs containing x by $\mathscr{D}_x$. Our next result shows that a *Maximal and Dense* DCS (MD-DCS) exists in $\mathscr{D}_x$.

Proposition **1** *For every $x \in A$, there exists a maximal element in $\mathscr{D}_x$ which is dense in A.*

The proof of this proposition is deferred to the Appendix.

Pick an arbitrary $x \in A$ and an MD-DCS D_x. All the following results until Claim 10 refer to x and D_x. Claims 3 to 11 have counterparts in Lavi et al. (2009) and Roberts (1979) and we follow their arguments closely. At certain places new constructions are required because of the special nature of our domain. We point this out at the appropriate places.

The *value difference set* for any pair of *distinct* allocations $y, z \in D_x$ is defined as follows.

$$P(y,z) = \{\alpha \in \mathbb{R}^n : \exists\ u \in U^n \text{ s.t. } u(y) - u(z) = \alpha \text{ and } F(u) = y\}.$$

The set $P(y,z)$ is non-empty since F is onto. The next claim shows that the positive quadrant starting from a point in $P(y,z)$ is always contained in $P(y,z)$.

Claim **3** *If $\alpha \in P(y,z)$, and $\delta > \mathbf{0} \in \mathbb{R}^n$, then $\alpha + \delta \in P(y,z)$, for all distinct $y, z \in D_x$.*

Proof: We are given that $\alpha \in P(y,z)$. By definition of $P(y,z)$, there exists $u \in U^n$ such that $F(u) = y$ and $u(y) - u(z) = \alpha$. Construct $u' \in U^n$ such that $u'_i(y_i) - u_i(y_i) = \delta_i > u'_i(w_i) - u_i(w_i), \forall w_i \neq y_i, z_i, \forall i \in N, w \in A$, where $\delta_i > 0$ and $u'_i(z_i) - u_i(z_i) = 0, \forall i \in N$. This function u' can be constructed in exactly the same manner as u''_i in Claim 2. By PAD-DC, $F(u') = y$, and by construction, $u'(y) - u'(z) = \alpha + \delta$. Hence $\alpha + \delta \in P(y,z)$, as needed. ■

Claim **4** *For every $\alpha, \epsilon \in \mathbb{R}^n$, $\epsilon > \mathbf{0}$, and for all $y, z \in D_x$,*
(a) $\alpha - \epsilon \in P(y,z) \Rightarrow -\alpha \notin P(z,y)$.
(b) $\alpha \notin P(y,z) \Rightarrow -\alpha \in P(z,y)$.

Proof: (a) Suppose, for contradiction, $-\alpha \in P(z,y)$. Then, there exists $u \in U^n$ such that $u(z) - u(y) = -\alpha$ and $F(u) = z$. Since $\alpha - \epsilon \in P(y,z)$, there exists u' such that $u'(y) - u'(z) = \alpha - \epsilon$ and $F(u') = y$. But $u'(y) - u'(z) < u(y) - u(z)$. This is a contradiction, since $F(u) \neq z$ according to Claim 2.

(b) [8] Let B_y and B_z be two disjoint open balls centered at y and z respectively, i.e., $B_y = \{k : |k - y| < \epsilon_1\}$, $B_z = \{k : |k - z| < \epsilon_2\}$, for some $\epsilon_1, \epsilon_2 > 0$. These exist since y and z do not match in any of their components. Choose a $u \in U^n$ such that $u(y) - u(z) = \alpha$ and $u_i(y_i) - u_i(w_i) > u^y_i(y_i) - u^y_i(w_i)$, for all $u^y \in F^{-1}(y)$, $w \in A \setminus (\{y\} \cup B_z)$, and $y_i \neq w_i, \forall i \in N$. We also want u to satisfy: $u_i(z_i) - u_i(w_i) > u^z_i(z_i) - u^z_i(w_i)$, for all $u^z \in F^{-1}(z)$, $w \in A \setminus (\{z\} \cup B_y)$, and $z_i \neq w_i, i \in N$. The function u can be constructed in exactly the same way as u''_i in Claim 2.

We now argue that $F(u) \in \{y, z\}$. If not, then $F(u)$ either belongs to $A \setminus (\{y\} \cup B_z)$ or $A \setminus (\{z\} \cup B_y)$. In both cases, Claim 2 and the earlier inequalities imply $F(u) \neq w$, which is

[8] This proof is constructive and inspired by the 'flexibility' condition of valuations in Carbajal et al. (2013).

a contradiction. Since we have shown $F(u) \in \{y, z\}$ and $u(y) - u(z) = \alpha \notin P(y, z)$, we have $F(u) = z$. Since $u(z) - u(y) = -\alpha$, it follows that $-\alpha \in P(z, y)$. ■

The next result shows the relationship between the sum of the interiors of two value difference sets with that of a third.

Claim **5** *For every $\alpha, \beta, \epsilon^\alpha, \epsilon^\beta \in \mathbb{R}^n$, $\epsilon^\alpha, \epsilon^\beta > \mathbf{0}$, and $\forall y, z, w \in D_x$,*

$$\alpha - \epsilon^\alpha \in P(y, z) \text{ and } \beta - \epsilon^\beta \in P(z, w) \;\Rightarrow\; \alpha + \beta - (\epsilon^\alpha + \epsilon^\beta)/2 \in P(y, w).$$

Proof: The proof proceeds in a similar way as in part (b) of Claim 4, by successive elimination of possible allocations. We start by constructing three pairwise disjoint balls B_y, B_z, B_w around y, z, w. This is possible since the points do not match in any of their components. Choose $u \in U^n$ such that,

$$u(y) - u(z) = \alpha - \epsilon^\alpha/2 \tag{4}$$

$$u(z) - u(w) = \beta - \epsilon^\beta/2 \tag{5}$$

$$\begin{aligned}
u_i(y_i) - u_i(l_i) &> u_i^y(y_i) - u_i^y(l_i), \\
&\qquad \forall u^y \in F^{-1}(y), l \in A \setminus (\{y\} \cup B_z \cup B_w), y_i \neq l_i, i \in N \\
u_i(z_i) - u_i(l_i) &> u_i^z(z_i) - u_i^z(l_i), \\
&\qquad \forall u^z \in F^{-1}(z), l \in A \setminus (\{z\} \cup B_w \cup B_y), z_i \neq l_i, i \in N \\
u_i(w_i) - u_i(l_i) &> u_i^w(w_i) - u_i^w(l_i), \\
&\qquad \forall u^w \in F^{-1}(w), l \in A \setminus (\{w\} \cup B_y \cup B_z), w_i \neq l_i, i \in N
\end{aligned}$$

Such a function u can be constructed using arguments similar to those in Claim 4(b). Also, we can conclude that $F(u) \in \{y, z, w\}$. Since $\alpha - \epsilon^\alpha \in P(y, z)$, Equation (4) and an application of Claim 2 yields $F(u) \neq z$. Similarly, since $\beta - \epsilon^\beta \in P(z, w)$, Equation (5) and Claim 2 gives $F(u) \neq w$. Hence $F(u) = y$. Thus, $\alpha + \beta - (\epsilon^\alpha + \epsilon^\beta)/2 = u(y) - u(w) \in P(y, w)$, as needed. ■

The *minimum translation factor* is defined as follows:

$$\gamma(y, z) = \inf\{p \in \mathbb{R} \mid p\mathbf{1} \in P(y, z)\}, \text{ for all distinct } y, z \in D_x.$$

Claim **6** *For all distinct $y, z \in D_x$, $\gamma(y, z)$ exists in $\mathbb{R}$.*

Proof: To show this, we need to show that the set $\{p \in \mathbb{R} \mid p\mathbf{1} \in P(y, z)\}$ is non-empty. Pick $u \in U^n$ such that $F(u) = y$, and let $p = \max_i\{u_i(y_i) - u_i(z_i)\}$. Increase all co-ordinates of $u(y)$ to obtain $u'(y)$ such that $u'_i(y_i) - u'_i(z_i) = p, \forall i$. Note that this possible since U is an ASV domain and the fact that both y and z are chosen from D_x. By PAD-DC, $F(u') = y$. Hence, the set $\{p \in \mathbb{R} \mid p\mathbf{1} \in P(y, z)\}$ is non-empty, i.e., $\gamma(y, z) < \infty$. Also $\{p \in \mathbb{R} \mid p\mathbf{1} \in P(y, z)\}$ is bounded from below, otherwise $P(z, y)$ would be empty (by Claim 4(b)), i.e., $\gamma(y, z) > -\infty$. Hence, $\gamma(y, z)$ is a real number. ■

Claim **7** *For all $y, z, w \in D_x$, the following holds:*
 (a) $\gamma(y,z) = -\gamma(z,y)$.
 (b) $\gamma(y,w) = \gamma(y,z) + \gamma(z,w)$.

Proof: (a) Let $\gamma(y,z) = p^*$. Thus for any $\epsilon > 0$, $(p^* + \epsilon - (\epsilon/2))\mathbf{1} \in P(y,z)$. By Claim 4, $(-p^* - \epsilon)\mathbf{1} \notin P(z,y)$. On the other hand, by definition of $\gamma(y,z)$, $(p^* - \epsilon)\mathbf{1} \notin P(y,z)$. Therefore, Claim 4 implies $(-p^* + \epsilon)\mathbf{1} \in P(z,y)$. Hence, $-p^* = \inf\{p \mid p\mathbf{1} \in P(z,y)\} = \gamma(z,y)$, as needed.

(b) Fix $\epsilon > 0$. We have $(\gamma(y,z)+\epsilon-(\epsilon/2))\mathbf{1} \in P(y,z)$, and $(\gamma(z,w)+\epsilon-(\epsilon/2))\mathbf{1} \in P(z,w)$. From Claim 5, we have $(\gamma(y,z)+\gamma(z,w)+2\epsilon-(\epsilon/2))\mathbf{1} \in P(y,w)$. Since ϵ is arbitrary, we take its limit to zero to obtain, $\gamma(y,w) \leq \gamma(y,z)+\gamma(z,w)$. Exchanging z and w, we get $\gamma(y,z) \leq \gamma(y,w)+\gamma(w,z)$. Since $\gamma(w,z) = -\gamma(z,w)$ (from part (a) above), we get, $\gamma(y,w) \geq \gamma(y,z) + \gamma(z,w)$. Hence, $\gamma(y,w) = \gamma(y,z) + \gamma(z,w)$, as needed. ■

Unlike the standard affine maximizer results (Roberts, 1979; Lavi et al., 2009) in finite setting, we require $\gamma(\cdot,\cdot)$ to satisfy a continuity property in ASV domain.

Claim **8** *The minimum translation factor $\gamma(y,z)$ is continuous in both arguments.*

Proof: The continuity of $\gamma(y,z)$ in z is a consequence of the assumption that the valuation functions in the ASV domain are continuous. Since $\gamma(y,z) = -\gamma(z,y)$, it is also continuous in y. ■

Define the translated set $C(y,z) = P(y,z) - \gamma(y,z)\mathbf{1} = \{\alpha - \gamma(y,z)\mathbf{1} \mid \alpha \in P(y,z)\}$. Denote the 'interior' of $C(y,z)$ by $\mathring{C}(y,z)$, i.e.,

$$\mathring{C}(y,z) = \{\alpha \in C(y,z) \mid \alpha - \epsilon \in C(y,z), \text{ for some } \epsilon > \mathbf{0}\}.$$

Claim **9** *$\mathring{C}(y,z) = \mathring{C}(w,l)$, for all $y, z, w, l \in D_x$, $y \neq z$ and $w \neq l$.*

Remark: Note that this result includes the cases, $\mathring{C}(y,z) = \mathring{C}(w,z) = \mathring{C}(w,y) = \mathring{C}(z,y)$. Therefore, Claim 9 holds even when y, z, w, l are not all distinct.

Proof: Claim 5 yields the following:

$$\mathring{P}(y,z) \subseteq \mathring{P}(y,l) - \beta, \quad \forall \beta \in \mathring{P}(z,l)$$
$$\mathring{P}(z,l) \subseteq \mathring{P}(y,l) - \alpha, \quad \forall \alpha \in \mathring{P}(y,z)$$

Substituting y for z, and w for y in the second equation above, we get:

$$\mathring{P}(y,l) \subseteq \mathring{P}(w,l) - \alpha, \quad \forall \alpha \in \mathring{P}(w,y)$$

This implies,

$$\mathring{P}(y,z) \subseteq \mathring{P}(w,l) - (\alpha + \beta), \text{ where } \alpha \in \mathring{P}(w,y), \beta \in \mathring{P}(z,l).$$

In particular, this is true for $\alpha = (\gamma(w,y) + \epsilon)\mathbf{1}$, and $\beta = (\gamma(z,l) + \epsilon)\mathbf{1}$, where $\epsilon > 0$. Since, ϵ can be made arbitrarily small, it follows that:

$$\begin{aligned}
\mathring{P}(y,z) &\subseteq \mathring{P}(w,l) - (\gamma(w,y) + \gamma(z,l))\mathbf{1} \\
&= \mathring{P}(w,l) - (\gamma(w,y) + \gamma(y,l) - \gamma(y,l) + \gamma(z,l))\mathbf{1} \\
&= \mathring{P}(w,l) - (\gamma(w,y) + \gamma(y,l) + \gamma(l,y) + \gamma(z,l))\mathbf{1} \\
&= \mathring{P}(w,l) - (\gamma(w,l) + \gamma(z,y))\mathbf{1} \\
\Rightarrow \quad \mathring{P}(y,z) - \gamma(y,z)\mathbf{1} &\subseteq \mathring{P}(w,l) - \gamma(w,l)\mathbf{1}
\end{aligned}$$

In the second and third equalities and in the final step above, we have used Claim 7. This proves $\mathring{C}(y,z) \subseteq \mathring{C}(w,l)$. Swapping the allocations y with w, and z with l and redoing the analysis, we conclude that $\mathring{C}(y,z) \supseteq \mathring{C}(w,l)$. Therefore $\mathring{C}(y,z) = \mathring{C}(w,l)$, as needed. ■

In view of this claim, we denote $C_x = \mathring{C}(y,z) = \mathring{C}(w,l)$. Our next claim shows that this set is also independent of x.

Claim **10** *$C_x = C_y, \ \forall\ x, y \in A$.*

Proof: Construct two points in $D_x \cap D_y$. Pick the first point z_1 such that all its components are different from the corresponding components of both x and y. This is possible to pick since both x and y have finite number of components and both D_x and D_y are maximal sets. The second point z_2 is picked such that all its components are different from the corresponding components of both x, y and z_1. The argument of existence follows similarly. Therefore, we conclude that $C_x = \mathring{C}(z_1, z_2) = C_y$, as needed. ■

In view of Claim 10, the subscript in C_x can be removed. The next claim shows that this set is convex.

Claim **11** *C is convex.*

Proof: Pick distinct $\alpha, \beta \in C$. We will show $(\alpha+\beta)/2 \in C$. Since C is open, it follows that C is convex. Fix distinct $y, z, w \in D$. By definition, $\alpha \in \mathring{P}(y,z) - \gamma(y,z)\mathbf{1}$ and $\beta \in \mathring{P}(z,w) - \gamma(z,w)\mathbf{1}$. Therefore, $\alpha+\beta \in (\mathring{P}(y,z) + \mathring{P}(z,w)) - (\gamma(y,z)\mathbf{1} + \gamma(z,w)\mathbf{1}) \subseteq \mathring{P}(y,w) - \gamma(y,w)\mathbf{1}$ (using Claims 5 and 7). Hence, $\alpha + \beta \in C$.

We show that if $\alpha \in C$, then $\alpha/2 \in C$. Suppose not, i.e., $\alpha/2 \notin int(C(w,z))$ for some $w, z \in D_x$ and $x \in A$. It can be either $\alpha/2 \notin C(w,z)$ or $\alpha/2 \in bd(C(w,z))$ [9]. Therefore, for all $\delta \geq \mathbf{0},\ \delta \neq \mathbf{0}$, we have $\alpha/2 - \delta \notin C(w,z)$. An immediate consequence of Claim 4 is $-\alpha/2 + \delta \in C(z,w)$. Since $\alpha \in C$, there exists $\epsilon^\alpha > \mathbf{0}$ such that $\alpha - \epsilon^\alpha \in C(y,z)$ for some $y, z \in D_x$. Since $-\alpha/2 + \delta \in C(z,w)$ and δ is arbitrary, we can choose $\delta = \epsilon^\alpha/4$ so that $-\alpha/2 + \epsilon^\alpha/2 - \epsilon^\alpha/4 \in C(z,w)$. Hence, we conclude from Claim 5 that $\alpha + (-\alpha/2) - \epsilon^\alpha/8 \in C(y,w)$, i.e., $\alpha/2 \in C$. But this is a contradiction. ■

Since $C = \mathring{C}(y,z) = \mathring{C}(z,y)$ is convex and $\mathring{C}(y,z) = (cl(-\mathring{C}(z,y)))^c$ (Claim 4) [10], it follows that C is a convex half-space. Moreover, $\mathbf{0} \notin C$. Applying the separating hyperplane theorem, it follows that there exists $w \in \mathbb{R}^n_{\geq 0} \setminus \{\mathbf{0}\}$ such that, $w^\top \alpha \geq 0$, for all $\alpha \in cl(C)$.

Construction of the $\kappa(\cdot)$ function Fix an arbitrary $x_0 \in A$.

$$\kappa(x) = \begin{cases} \gamma(x_0, x) \cdot w^\top \mathbf{1} & \text{if } x_0 \in D_x \setminus \{x\} \\ \lim_{x_n \to x_0, x_n \in D_x, \forall n}\ \gamma(x_n, x) \cdot w^\top \mathbf{1} & \text{if } x_0 \notin D_x \\ 0 & \text{if } x = x_0 \end{cases} \tag{6}$$

This function is well defined since $\gamma(y,z)$ is continuous in both arguments (Claim 8) and D_x is dense in A. Hence every allocation $x_0 \in A$ is either an element of D_x or a limit point of D_x. Therefore,

[9] $bd(S)$ denotes the boundary of S.

[10] $cl(S)$ and S^c denote the closure and complement of S respectively.

if $F(u) = x$, we have $w^\top(u(x) - u(y) - \gamma(x, y)\mathbf{1}) \geq 0$ for all $y \in D_x$. We consider two cases. If $x_0 \in D_x$,

$$\begin{aligned} & w^\top(u(x) - u(y) - (\gamma(x, x_0) + \gamma(x_0, y))\mathbf{1}) \geq 0 \\ \Rightarrow\quad & w^\top u(x) + \gamma(x_0, x)w^\top \mathbf{1} \geq w^\top u(y) + \gamma(x_0, y)w^\top \mathbf{1} \\ \Rightarrow\quad & w^\top u(x) + \kappa(x) \geq w^\top u(y) + \kappa(y). \end{aligned}$$

If $x_0 \notin D_x$,

$$\begin{aligned} & w^\top(u(x) - u(y) - (\gamma(x, x_n) + \gamma(x_n, y))\mathbf{1}) \geq 0, \text{ for all } x_n \in D_x \\ \Rightarrow\quad & w^\top u(x) + \gamma(x_n, x)w^\top \mathbf{1} \geq w^\top u(y) + \gamma(x_n, y)w^\top \mathbf{1} \\ \Rightarrow\quad & w^\top u(x) + \lim_{x_n \to x_0, x_n \in D_x, \forall n} \gamma(x_n, x)w^\top \mathbf{1} \geq w^\top u(y) + \lim_{x_n \to x_0, x_n \in D_x, \forall n} \gamma(x_n, y)w^\top \mathbf{1} \\ \Rightarrow\quad & w^\top u(x) + \kappa(x) \geq w^\top u(y) + \kappa(y). \end{aligned}$$

Hence we have shown that if $F(u) = x$,

$$w^\top u(x) + \kappa(x) \geq w^\top u(y) + \kappa(y), \forall y \in D_x. \tag{7}$$

Since D_x is dense in A, each allocation $z \in A$ is either a member of D_x or its limit point. Claim 8 implies that κ is a continuous function. Let $\{y_n\} \to z$ where $y_n \in D_x, \forall n$. Thus Equation (7) holds for every y_n. Taking limits we conclude that,

$$w^\top u(x) + \kappa(x) \geq w^\top u(z) + \kappa(z), \forall z \in A.$$

This concludes the proof. *Q.E.D.*

5 Revenue Equivalence

In this section, we show that the SCF F also satisfies revenue equivalence. This is an observation based on the definition of revenue equivalence and a standard result (Rockafellar, 1997). The following definition of revenue equivalence is standard in literature.

Definition **8 (Revenue Equivalence)** *An SCF F satisfies* revenue equivalence *if for any two payment rules p and p' that make F strategyproof, there exist functions $\alpha_i : U^{n-1} \to \mathbb{R}$, such that,*

$$p_i(u_i, u_{-i}) = p'_i(u_i, u_{-i}) + \alpha_i(u_{-i}),\ \forall u_i \in U, \forall u_{-i} \in U^{n-1}, \forall i \in N.$$

This essentially ensures that any two payment rules that make F strategyproof differ only by a factor which is independent of individual agents' valuations. The following result shows that convexity and linearity of valuations is sufficient for a strategyproof SCF to satisfy revenue equivalence.

Theorem **2 (Rockafellar (1997); Krishna and Maenner (2001))** *If the type space is convex and the valuations are linear in type, then a strategyproof SCF satisfies revenue equivalence.*

In our setting, since U is the ASV domain, it is clearly convex. Also the types of the agents are their valuations, which implies, trivially, that the valuations are linear in its type. Hence, the above result on revenue equivalence holds for F and we conclude that F satisfies revenue equivalence.

6 A More General Result

Our affine maximizer result rests on the fact that the allocation space is rich enough to admit a maximal and dense subset D_x for each allocation x where selfish valuations generate *sufficiently* rich value differences. A generalization of this result yields the following theorem, the proof of which is omitted here.

Theorem **3 (Affine Maximizers for Selfish Valuations)** *Let the space of allocations A be separable into components for each agent and be compact. If there exists at least one collection of MD-DCSs $\{D_x \mid x \in A\}$ such that for all $x, y \in A$, $|D_x \cap D_y| \geq 2$, then an* onto, ANB and strategyproof *SCF $F : U^n \to A$ is an affine maximizer.*

7 Conclusions

From the results of this paper, it appears that in order to prove that strategyproof SCFs are affine maximizers, perhaps a certain amount of richness is required, which may be split across the set of valuations and allocations. We considered a sub-domain of the unrestricted valuations, namely the domain of selfish valuations but a "rich" set of allocations to prove our affine maximizer result.

An interesting open question is whether our results extend to more restricted domains, e.g., domains with increasing valuations.

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Appendix

Proof of Claim 1

Proof: Pick an arbitrary onto affine maximizer and a tie-breaking rule t, i.e., $t : A \to \mathbb{R}$ is an *injective* mapping. The affine maximizer together with the tie-breaking rule produces an SCF F in a natural way: at every valuation profile the tie-breaking rule picks a unique allocation from the set of affine maximizers. We can use standard arguments in the literature to show that this SCF is strategyproof. In particular, the following payment rule implements this SCF. For every $i \in N$:

$$p_i(u_i, u_{-i}) = \begin{cases} \frac{1}{w_i}\left(\sum_{j \neq i} w_j u_j(F(u)) + \kappa(F(u)) + h_i(u_{-i})\right), & w_i > 0 \\ 0 & w_i = 0 \end{cases} \tag{8}$$

Where h_i is any arbitrary function of the valuations of all the agents except agent i. We claim that F is ANB.

Assume for contradiction that F is bossy. Therefore, there exists some agent i such that $x = (x_i, x_{-i}) = F(u_i, u_{-i}) \in \mathcal{A}(u_i, u_{-i})$, where $\mathcal{A}(u_i, u_{-i})$ is the set of affine maximizers when the valuation profile is (u_i, u_{-i}), and $y = (x_i, y_{-i}) = F(u'_i, u_{-i}) \in \mathcal{A}(u'_i, u_{-i})$. Now, from the definition of affine maximizer when the valuation profile is (u_i, u_{-i}), we have:

$$w_i u_i(x_i) + \sum_{j \neq i} w_j u_j(x_j) + \kappa(x) \geq w_i u_i(x_i) + \sum_{j \neq i} w_j u_j(y_j) + \kappa(y) \tag{9}$$

$$\Rightarrow \sum_{j \neq i} w_j u_j(x_j) + \kappa(x) \geq \sum_{j \neq i} w_j u_j(y_j) + \kappa(y)$$

Similarly, for valuation profile (u'_i, u_{-i}), we have:

$$w_i u'_i(x_i) + \sum_{j \neq i} w_j u_j(y_j) + \kappa(y) \geq w_i u'_i(x_i) + \sum_{j \neq i} w_j u_j(x_j) + \kappa(x)$$

$$\Rightarrow \sum_{j \neq i} w_j u_j(y_j) + \kappa(y) \geq \sum_{j \neq i} w_j u_j(x_j) + \kappa(x)$$

Hence, we get:

$$\sum_{j \neq i} w_j u_j(y_j) + \kappa(y) = \sum_{j \neq i} w_j u_j(x_j) + \kappa(x) \tag{10}$$

Thus $x, y \in \mathcal{A}(u'_i, u_{-i})$ and $x, y \in \mathcal{A}(u_i, u_{-i})$. However, this contradicts the definition of the tie-breaking rule t. If x is chosen over y at the valuation profile (u_i, u_{-i}), then y cannot be chosen over x at the valuation profile (u'_i, u_{-i}). ■

Proof of Proposition 1

Recall that $\mathscr{D}_x$ is the set of all DCSs that contain the allocation x. Define a partial order $\succsim$ on $\mathscr{D}_x$ with respect to the set containment relation as follows. For all $E_x, E'_x \in \mathscr{D}_x$, $E_x \succsim E'_x \Leftrightarrow E'_x \subseteq E_x$.

Lemma **2** *$(\mathscr{D}_x, \succsim)$ has a maximal element.*

Proof: Consider an arbitrary *chain*, i.e., a linearly ordered subclass , say $(\mathcal{B}, \succsim|_{\mathcal{B}})$ within $(\mathscr{D}_x, \succsim)$. By definition, $\mathcal{B} \subseteq \mathscr{D}_x$ and $\succsim|_{\mathcal{B}}$ is a restriction of $\succsim$ to $\mathcal{B}$, which is complete and anti-symmetric. Consider $E^{\mathcal{B}} = \cup_{E \in \mathcal{B}} E$.

We claim that $E^{\mathcal{B}} \in \mathscr{D}_x$, i.e., $E^{\mathcal{B}}$ is a DCS. Suppose not. Then $\exists\, y, z \in E^{\mathcal{B}}, y \neq z$ such that y and z are identical in at least one component. By definition of $E^{\mathcal{B}}$ and considering the fact that $(\mathcal{B}, \succsim|_{\mathcal{B}})$ is a chain, it must be true that $\exists E \in \mathcal{B}$ such that $y, z \in E$ (every element in a set E continues to be members of its supersets, hence if y is a member of a set E_1 and z a member of $E_2 \supseteq E_1$, then $y \in E_2$, then E_2 is our chosen E). This shows that E is not a DCS. This contradicts the fact $\mathcal{B} \subseteq \mathscr{D}_x$, since all elements of $\mathcal{B}$ must be DCSs.

Notice that $E^{\mathcal{B}} \succsim E, \forall E \in \mathcal{B}$. Hence every chain in $(\mathscr{D}_x, \succsim)$ has an upper bound in $\mathscr{D}_x$, and this is true for every $\mathcal{B}$. By Zorn's lemma, $(\mathscr{D}_x, \succsim)$ has a maximal element. ■

Lemma **3** *There exists a $D \in \mathscr{D}_x$ such that D is dense in A.*

Proof: The proof is constructive. We first prove a claim that is used in the construction. Define a *grid* G_x generated by an allocation x as the set of points in A that agree with x in at least one component, i.e.,

$$G_x := \cup_{i \in N} \{y \in A \mid y_i = x_i\}.$$

A grid splits the space A into several cells. A *cell profile* is the collection of allocations except the grid, i.e., $A \setminus G_x$. Since x has finite number of components n, the grid G_x divides $A \setminus G_x$ into a finite number of disjoint *connected* [11] sets of allocations. Let us denote them by C_k's, i.e., $A \setminus G_x = \cup_{k=1}^{K} C_k$, where K is some finite number. Let us denote the interiors of C_k by $\mathring{C}_k$. Hence $int(A \setminus G_x) = \cup_{k=1}^{K} \mathring{C}_k$ is an open set. We will call this set as the *open cell profile* induced by the allocation x and $\mathring{C}_k$'s as the open cells corresponding to the cell profile $A \setminus G_x$. Denote the *center of mass* of cell $\mathring{C}_k$ as $\mathrm{cm}(\mathring{C}_k)$. For the special structure of the cells induced by the grid, the center of mass is simply represented by the average of the co-ordinates of the vertices of $\mathring{C}_k$. Define the radius of the largest sphere centered at $\mathrm{cm}(\mathring{C}_k)$ that can be inscribed in $\mathring{C}_k$ as:

$$r(\mathring{C}_k) := \sup\{r \in \mathbb{R}_{\geq 0} \mid B(\mathrm{cm}(\mathring{C}_k), r) \subseteq \mathring{C}_k\},$$

where $B(c, r) := \{x \in A : ||x - c|| < r\}$ is an open ball of radius r centered at c.

Claim **12** *For each grid G_x induced by $x \in A$, there exists $C_{G_x} \subseteq A$ such that,*

(i) each open cell $\mathring{C}_k$ of $A \setminus G_x$ has exactly one point of C_{G_x} within $B(cm(\mathring{C}_k), r(\mathring{C}_k)/2)$, and

(ii) C_{G_x} is a DCS.

Proof: Consider the collection of open balls centered at $\mathrm{cm}(\mathring{C}_k)$ with radii $r(\mathring{C}_k)/2)$, i.e., $\cup_{k=1}^{K} B(\mathrm{cm}(\mathring{C}_k), r(\mathring{C}_k)/2)$. Define a random experiment which selects one point uniformly at random from each of these open balls $B(\mathrm{cm}(\mathring{C}_k), r(\mathring{C}_k)/2)$, independently from every other ball.

[11] Two subsets A and B of a metric space X are separated if $A \cap \bar{B} = \bar{A} \cap B = \phi$. A set $C \subseteq X$ is connected if C is not an union of two nonempty separated sets.

Clearly, it is a measure zero event to pick two points which have at least one identical component. Hence, there exists at least one configuration which has distinct components and hence this is the desired DCS C_{G_x} as claimed. ■

With the help of the above claim, we will now construct a set $D \in \mathscr{D}_x$ that is dense in A using the following algorithm.

(1) We start with x and call $D_1 = \{x\}$. Construct a grid G_x and consider the open cell profile $int(A \setminus G_x)$. Pick the DCS C_{G_x} as constructed in Claim 12. Notice that by construction the set of allocations $\{x\} \cup C_{G_x} =: D_2$ is also a DCS.

(2) For each allocation $z \in D_2$, we can construct grids G_z and their union is denoted by $G_{D_2} = \cup_{z \in D_2} G_z$. The open cell profile $int(A \setminus (G_{D_2} \cup G_{D_1}))$ has a finite number of cells. Construct a DCS $C_{G_{D_1} \cup G_{D_2}}$ in similar lines of Claim 12. Denote $\{x\} \cup C_{G_{D_1} \cup G_{D_2}} =: D_3$

(3) Construct $D_4, D_5, \ldots$ in a similar way.

Figure 2 illustrates an example where the steps of the construction with one object ($m = 1$) and 3 agents ($n = 3$) are explained. The simplex (depicted by the plane described by the slant parallel lines) is the set of allocations A. The dashed lines show the grid structure (under construction) G_x and the simplex except the grid is the cell profile. It also shows one point z which is chosen from an open ball centered at the center of mass of cell C_k.

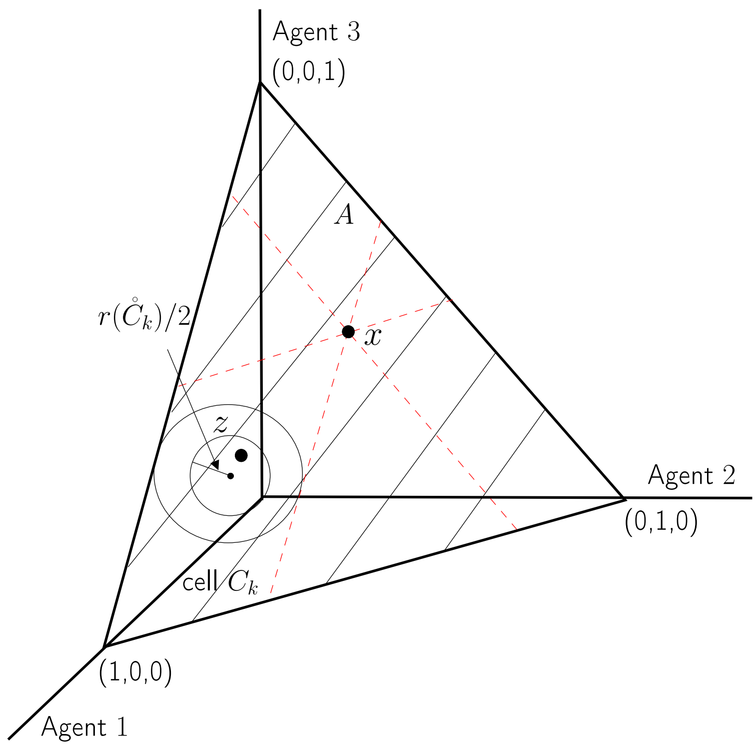

Figure 2: Illustration of the construction of D_l's.

We claim that $D := \cup_{l=1}^{\infty} D_l$ is dense in A. Clearly $D \in \mathscr{D}_x$, i.e., a DCS, by construction. To show that it is dense in A, pick an arbitrary point $p \in A$. If $p \in D$, we have nothing to prove. Let

$p \notin D$. The cell diameter (the maximum distance between any two points in the cell) was initially finite when we constructed the first open cell profile induced by the grid G_x.

(a) After that, in every step l, the maximum distance between any point in the cell $\mathring{C}_k$ and the newly added point z to the constructed DCS D_l can be at most 3/4-th of the diameter dia_l of that cell (1/2 because of choosing the center of mass of the cell and 1/4-th for choosing the radius $r(\mathring{C}_k)/2$).

(b) Also, since the cell at step l is induced by a grid $G_{z'}$ where z' was chosen in a similar fashion in step $(l-1)$, the diameter dia_l is at most 3/4-th of the earlier diameter dia_{l-1}.

Hence, there exists at least one new point z (in some cell) in D_l from which the distance of p reduces by a constant fraction of the earlier diameter, i.e., $||z-p|| \leq \frac{3}{4}dia_l$ (by (a)) $\leq \frac{3}{4} \cdot \frac{3}{4}dia_{l-1}$ (by (b)) $= \frac{3}{4} \cdot \left(\frac{3}{4}\right)^l dia_0$. Since $dia_0 < \infty$, it is clear that $||z-p||$ will be smaller than any chosen $\epsilon > 0$ for l larger than a large enough number of iterations $N_{\epsilon,p}$. Therefore, p is a limit point of D. Hence D is dense in A. This concludes the proof. ∎

Pick a $D \in \mathscr{D}_x$ dense in A. Define, $\chi_D = \{E_x \in \mathscr{D}_x \mid E_x \succsim D\}$. It is clear that every element of χ_D is dense in A. Also $D \in \chi_D$, therefore, $\chi_D \neq \emptyset$.

Lemma **4** $(\chi_D, \succsim|_{\chi_D})$ *has a maximal element.*

Proof: The argument of the proof of this lemma is similar to Lemma 2. Consider any chain $\mathcal{B} \subseteq \chi_D$. Then $\cup_{E_x \in \mathcal{B}} E_x \succsim E_x, \forall E_x \in \mathcal{B}$ and is, therefore, an upper bound on $\mathcal{B}$. However, by definition, $D \subseteq E_x, \forall E_x \in \mathcal{B} \subseteq \chi_D$. Hence, $D \subseteq \cup_{E_x \in \mathcal{B}} E_x$. This implies that $\cup_{E_x \in \mathcal{B}} E_x \in \chi_D$.

So, every chain in $(\chi_D, \succsim|_{\chi_D})$ has been shown to have an upper bound in $(\chi_D, \succsim|_{\chi_D})$. Thus, by Zorn's lemma, the conclusion follows. ∎

We show that this maximal element is a maximal element in the set of all DCSs.

Lemma **5** *Every maximal element of* $(\chi_D, \succsim|_{\chi_D})$ *is a maximal element of* $(\mathscr{D}_x, \succsim)$.

Proof: Suppose not. Fix a maximal element θ_{χ_D} of $(\chi_D, \succsim|_{\chi_D})$ that is not maximal in $(\mathscr{D}_x, \succsim)$. Therefore, there exists some $\theta \in \mathscr{D}_x$ such that $\theta \succ \theta_{\chi_D}$. By definition, $\theta_{\chi_D} \succsim D$. By transitivity of order relation $\theta \succ D$, which implies $\theta \in \chi_D$. This contradicts the maximality of θ_{χ_D} in $(\chi_D, \succsim|_{\chi_D})$. ∎

Hence, there exists a maximal element of $\mathscr{D}_x$ which is dense in A, as needed.