CS711: Game Theory and Mechanism Design

Problem Set 2

October 9, 2018

Que 1. [Easy] Let the number of alternatives be m. Find the number of single-peaked strict preference orderings with respect to $<$ (an exogenously given ordering of alternatives). [Hint: for a fixed peak, how many strict orderings are possible in the single-peaked domain?]

Que 2. [Easy] The Borda and Condorcet Methods The following example was presented by Condorcet, in a critique of the Borda Method. A committee composed of 81 members is to choose a winner from among three candidates, A, B, and C. The rankings of the committee members appear in the following figure :

No. of voters	First candidate	Second candidate	Third candidate
30	*A*	*B*	*C*
1	*A*	*C*	*B*
29	*B*	*A*	*C*
10	*B*	*C*	*A*
10	*C*	*A*	*B*
1	*C*	*B*	*A*

a. Is there a Condorcet winner? If yes, who is the Condorcet winner?

b. What is the Borda ranking, and who is the Borda winner?

c. Based on your answers above, what is Condorcet's critique of the Borda Method?

Que 3. [Easy] Consider a two agent model with three alternatives $\{a, b, c\}$. Figure shows two preference profiles of preferences. Suppose $f(P_1, P_2) = a$. Show that if f is strategyproof then $f(P_1, P_2) = b$. You are allowed to use the result that for any preference profile $(\bar{P}_1, \bar{P}_2)$, $f(\bar{P}_1, \bar{P}_2) \in \{\bar{P}_1(1), \bar{P}_2(1)\}$ (but do not use any other result from the lectures).

P_1	P_2	P_1'	P_2'
a	*c*	*b*	*c*
b	*b*	*a*	*a*
c	*a*	*c*	*b*

Que 4. [Easy] To Prove: A dictatorial social welfare function satisfies the properties of unanimity and independence of irrelevant alternatives.

Que 5. [Easy] To Prove: A dictatorial social choice function is monotonic.

Que 6. [Easy] To Prove: When no of choices,|A|=2, the simple majority rule satisfies the properties of unanimity and independence of irrelevant alternatives(IIA).

Que 7. [Easy]To prove: When no. of choices,|A| = 2 and when no. of individual making a choice,n is odd, the social choice function defined by majority rule is monotonic and non-dictatorial.

Que 8. [Easy]Consider the single-peaked domain model. A social choice function f is manipulable by a group of agents $K \subseteq N$ is for some preference profile (P_K, P_{-K}) there exists some preference profile P'_K of agents in K such that $f(P'_K, P_{-K})P_i f(P_K, P_{-K})$ for all $i \in K$. A social choice function f is **group strategy-proof** if it cannot be manipulated by any group of agents. Is the median voter SCF group strategy-proof?

Que 9. [Easy] Is a social choice function that is implementable in dominant strategies necessarily implementable in Nash Equilibrium?

Que 10. [Easy] Suppose there are k identical copies of a good and $n > k$ bidders. Suppose also that each bidder can receive at most one good. What is the analog of the second-price auction? Prove that your auction is DSIC.

Que 11. [Moderate] A committee comprised of 15 members is called upon to rank three colors: red, blue, and yellow, from most preferred to least preferred. The committee members simultaneously announce their strict preference relations over the three colors. If red is the preferred color of at least five members of the committee, red is determined to be the prettiest color. Otherwise, if blue is the preferred color of at least five members of the committee, blue is determined to be the prettiest color. Otherwise, yellow is determined to be the prettiest color. Once the prettiest color is determined, the remaining two colors are then ranked by the simple majority rule.

1. Is the social welfare function described here dictatorial? Justify your answer.
2. Does the social welfare function described here satisfy the unanimity property? Justify your answer.
3. Does the social welfare function described here satisfy the independence of irrelevant alternatives property? Justify your answers.

Que 12. [Moderate] Is the social welfare functions in the above question is monotonic? Justify your answer.

Que 13. [Moderate] Does every monotonic social welfare function satisfy the unanimity property? Justify your answer.

Que 14. [Moderate] In the private divisible good allocation model, discuss a social choice function that is strategy-proof and Pareto efficient but not anonymous.

Que 15. [Moderate] For each of the following relations, determine whether it is complete, reflexive, ir-reflexive, or transitive, and use this to determine whether it is a preference relation or a strict preference relation.
(a) A is the set of all subsets of some set S, and $a \succeq b$ if and only if b is a subset of a.
(b) A is the set of all natural numbers, and $a \succeq b$ if and only if b is a divisor of a (i.e., $a = bq$ for some integer q).
(c) A is the set of all 26 letters in the English alphabet, and $a \succeq b$ if and only if ab is a standard word in English (where a is the first letter of the word, and b the second letter of the word).
(d) A is the set of all natural numbers, and $a \succeq b$ if and only if $a * b = 30$.
(e) A is the set of all human beings, past and present, and $b \succeq a$ if and only if a is a descendant of b (meaning a child, grandchild, great-grandchild, etc).
(f) A is the set of people living in a particular neighborhood, and $a \succeq b$ if and only if a likes b.

Que 16. [Moderate] The following electoral method is used to choose the mayor of Whoville: Every resident ranks the candidates from most preferred to least preferred, and places this ranked list in a ballot box. Each candidate receives a number of points equal to the number of residents who prefer him to all the other candidates. The candidate who thus amasses the greatest number of points wins the election. If two or more candidates are tied for first place in the number of points, the winner of the election is the candidate among them whose social security number is smallest. Assume there are at least three candidates.
(a) Show by example that this electoral method is not monotonic.
(b) Show by example that this electoral method is not manipulable.

Que 17. [Moderate] Suppose that $|A| \geq 3$, and let F be a social welfare function satisfying the properties of unanimity and independence of irrelevant alternatives. What are all the decisive coalitions?

Que 18. [Moderate] Consider the unanimous $SCFf$ defined as follows. If $P_1(1) = \ldots = P_n(1) = a$, then $f(P_1, \ldots, P_n) = a$. Else, $f(P_1, \ldots, P_n) = b$ for some alternative $b \in A$. In other words, f satisfies unanimity wherever possible and picks a "status-quo" alternative b otherwise. Argue how f can be manipulated if there are at least three alternatives?

Que 19. [Moderate] Show that, for any I, when X contains two elements (say, $X = x_1, x_2$), then any majority voting social choice function [i.e., a social choice function that always chooses alternative x_i, if more agents prefer x_i over x_j than prefer x_j over x_i(it may select either x_1 or x_2 if the number of agents preferring x_1 over x_2 equals the number preferring x_2 over x_1)] is truthfully implementable in dominant strategies.

Que 20. [Moderate] Prove that every median voter social choice function is strategy- proof.

Que 21. [Difficult] Assume the number of voters is odd. We consider only preference profiles, which admit a Condorcet winner. Show that the social choice function which always picks the Condorcet winner is strategy-proof.

Que 22. [Difficult] Consider the single-peaked domain model. A social choice function f is manipulable by a group of agents $K \subseteq N$ if for some preference profile (P_K, P_{-K}) there exists some preference profile P'_K of agents in K such that $f(P'_K, P'_{-K})P_i f(P_K, P_{-K})$ for all $i \in K$. A social choice function f is group strategyproof if cannot be manipulated by any group of agents. Is the median voter SCF group strategyproof? Explain your answer.

Que 23. [Difficult] Let A be a finite set of alternatives and $f : P_n \to A$ be a social choice function that is unanimous and strategy proof. Suppose $|A| > 3$.
Now, consider another social choice function $g : P^2 \to A$ defined as follows. The SCF g only considers profiles of two agents, denote these two agents as 1 and 2. For any $(P_1, P_2) \in P^2$, let

$$g(P_1, P_2) = f(P_1, P_2, P_1, P_1, ..., P_1) \ ,$$

i.e., the outcome of g at (P_1, P_2) coincides with the outcome of f at the profile where agents 1 and 2 have types P_1 and P_2 respectively, and all other agents have type P_1 .
Show that g is a dictatorial SCF. [Hint: you may use the Gibbard-Satterthwaite theorem.]

Que 24. [Difficult] To Prove:(Arrow's Impossibility Theorem) If $|A| \geq 3$, there does not exist a social welfare function satisfying the properties of unanimity, independence of irrelevant alternatives, and non-dictatorship.

Que 25. [Difficult] Show that there is at most one dictator in every social welfare function F : if i is a dictator in F , and j is also a dictator in F , then $i = j$.

Que 26. [Difficult] To Prove: let F be a social welfare function satisfying the unanimity property. Then for every $a, b \in A$ the coalition N is decisive for a over b, and the empty coalition ϕ is not decisive for a over b.

Que 27. [Difficult] Let X be a set of projects. A social choice function chooses a non-empty subset of projects. Agent i has a linear ordering P_i over the set of projects X. Agent i evaluates subsets of projects by extending P_i in the following manner, for any pair of subsets of projects $S, T \subseteq X, S$ is preferred to T if the highest ranked project in S (according to P_i) is better than the highest ranked project in T, if thesse two projects are the same, then S and T are indifferent. Suppose $\mid X \mid \geq 2$. Will the Gibbard-Satterthwaite result apply here? Discuss your answer.

Que 28. [Difficult] A seller is selling an object to an agent whose value (type) for the object lies in the interval $I \equiv [0, 1]$. The seller uses an allocation rule $f :\to [0, 1]$ and a payment rule $p : I \to \mathbb{R}$. Denote the mechanism $(f, p)asM$. Fix an $\epsilon \in (0, 1]$. The mechanism M satisfies only a subset of incentive constraints: for every $t \in I$ and for every $s \in I$ such that $\mid s - t \mid \leq \epsilon$,

$$tf(t) - p(t) \geq tf(s) - p(s).$$

Show that M is dominant strategy incentive compatible.

Que 29. [Difficult]Prove that Truth-telling is a dominant strategy under any Groves mechanism.

Que 30. [Difficult] Suppose f is strategy-proof and peaks-only. Then prove that, at any preference profile P and for every agent $i \in N$,

$$f(P) = med(f(P^0, P_{-i}), P_i(1), f(P^1, P_{-i})).$$