CS711
Game Theory and Mechanism Design

Problem Set 1

August 13, 2018

Que 1. [Easy] William and Henry are participants in a televised game show, seated in separate booths with no possibility of communicating with each other. Each one of them is asked to submit, in a sealed envelope, one of the following two requests (requests that are guaranteed to be honored):

- Give me $1,000
- Give the other participant $4,000

1. Describe this situation as strategic-form game(Normal Form game).
2. List out the dominant strategies for both of them if any?
3. What is the equilibrium for this game?
4. What will be the best strategy for players, and why?

Que 2. [Easy] In each of the following games, where Player I is the row player and Player II is the column player. List down the strictly dominated strategies, weekly dominant strategies. Perform iterated elimination of strictly dominated strategies and verify whether it yields a single strategy vector when completed. If so, what is that vector? Verify that it is the only Nash equilibrium of the game.

1.

	L	R
H	4,2	0,1
T	1,1	3,3

2.

	L	R
H	1,3	2,3
T	0,4	0,2

3.

	A	B	C
D	1,0	3,0	2,1
E	3,1	0,1	1,2
F	2,1	1,6	0,2

4.

	A	B	C
D	40,8	30,6	50,1
E	30,6	20,9	50,1
F	20,5	40,2	50,0

5.

	A	B	C
D	6,6	8,20	0,8
E	10,0	5,5	2,8
F	8,0	20,0	4,4

Que 3. [Easy] Find the equilibrium in the following game.

	a	b	c	d
u	$0,0$	$-1,1$	$1,1$	$0,-1$
v	$1,-1$	$1,0$	$0,1$	$0,0$
w	$0,1$	$-1,-1$	$1,0$	$1,-1$
x	$-1,1$	$0,-1$	$-1,1$	$0,0$
y	$1,1$	$0,0$	$-1,-1$	$0,0$

Que 4. [Easy] Find the equilibria of the following three-player game (Player I chooses row T, C, or B, Player II chooses a column L, M, or R, and Player III chooses matrix P or Q).

	L	*M*	*R*
T	3, 10, 8	8, 14, 6	4, 12, 7
C	4, 7, 2	5, 5, 2	2, 2, 8
B	3, –5, 0	0, 3, 4	–3, 5, 0

P

	L	*M*	*R*
T	4, 9, 3	7, 8, 10	5, 7, –1
C	3, 4, 5	17, 3, 12	3, 5, 2
B	9, 7, 2	20, 0, 13	0, 15, 0

Q

Que 5. [Easy] For each of the following games, where Player I is the row player and Player II is the column player:

	L	*R*
T	5, 5	0, 8
B	8, 0	1, 1

Game A

	L	*R*
T	9, 5	10, 4
B	8, 4	15, 6

Game B

	L	*R*
T	5, 16	15, 8
B	16, 7	8, 15

Game C

	L	*R*
T	8, 3	10, 1
B	6, –6	3, 5

Game D

	L	*R*
T	4, 12	5, 10
B	3, 16	6, 22

Game E

	L	*R*
T	2, 2	3, 3
B	4, 0	2, –2

Game F

	L	*R*
T	15, 3	15, 10
B	15, 4	15, 7

Game G

1. Find all the equilibria in mixed strategies, and all the equilibrium payoffs.
2. Find each player's maxmin strategy.
3. What strategy would you advise each player to use in the game?

Que 6. [Easy] What will be a "reasonable" outcome of the game mentioned on slide 2 of lecture 1(July 30)?

Que 7. [Easy] Prove theorem from slide 8 of lecture 2(Aug 2) if the length of the game is infinite.

Que 8. [Easy] Prove: If $s_i^\star$ is a dominant strategy for player i, then it is a max-min strategy for player i as well, for all $i \in N$. Such a strategy is a best response of player i to any strategy profile of the other players(complete the proof discussed in the class).

Que 9. [Easy] Prove: If every player $i \in N$ has a strictly dominant strategy $s_i^\star$, then the strategy profile$(s_1^\star, ..., s_n^\star)$ is the unique equilibrium point of the game and also the unique profile of max-min strategies.

Que 10. [Easy] Elimination of dominated strategy of player j may increase the maxmin value of player $i \neq j$. Find an example where it happens.

Que 11. [Moderate] Prove: Consider an NFG $G = \langle N, (S_i)_{i \in N}, (u_i)_{i \in N} \rangle$, and let $\hat{G} = \langle N, (\hat{S}_i)_{i \in N}, (u_i)_{i \in N} \rangle$ be the game derived from G through elimination of some strategies, i.e.,$\hat{S}_i \subseteq S_i$, $\forall i \in N$. If $s^\star$ is a PSNE in G, and if $s_i^\star \in \hat{S}_i$ for every $i \in N$, then $s^\star$ is an equilibrium in $\hat{G}$.

Que 12. [Moderate] "The outcome of every play of the game of chess is either a victory for White, a victory for Black, or a draw." Is that statement equivalent to the result of theorem given by Von Neumann that we saw in class? Justify your answer.

Que 13. [Moderate] (True or False) "If a player has a dominant strategy in a simultaneous-move game, then she is sure to get her best possible outcome in any Nash equilibrium of the game." Explain your answer and give an example of a game that illustrates your answer.

Que 14. [Moderate] You and your sister have just inherited \$3M that needs to be split between the two of you. The rules are the same as above (offer, counteroffer, and final offer) except that each period, \$1M is removed from the total (each round of negotiation costs \$1M in lawyers fees). Further, assume that both you and your sister value future payments just as much as current payments (i.e. no discount factor). Calculate the Nash equilibrium for this game.

Que 15. [Moderate] (Provide an example of a 2-player game with strategy set $[0, \infty)$ for either player and payoffs continuous in the strategy profile, such that no strategy survives iterated deletion of strictly dominated strategies ($S^\infty = \phi$), but the set of strategies remaining at every stage is nonempty ($S_k \neq \phi$ for k = 1, 2, . . .).

Que 16. [Moderate] Rock Paper Scissors.

1. The game, "Rock, Paper, Scissors," which you have played since grade school, can be analyzed in this framework. In case you have forgotten, two players simultaneously choose to play "Rock", "Paper" or "Scissors". Rock beats Scissors, Scissors beats Paper, and Paper beats Rock. Suppose the "winner" of each game wins one point while the loser loses a point. Write down the payoff matrix that defines this game and find all pure or mixed strategy Nash equilibrium's.

2. Perturbed Rock Paper Scissors.
   Now suppose that the loser of each round loses two points instead of one. Write down the payoff matrix that defines this perturbed Rock Paper Scissors game and find all pure or mixed strategy Nash equilibrium.

Que 17. [Moderate] Find a game that has at least one equilibrium, but in which iterative elimination of dominated strategies yields a game with no equilibria.

Que 18. [Moderate] In the classical Neighboring Kingdoms' Dilemma game, each of two players had two possible actions, agriculture (A) and defense (D). The monetary payoff to each action profile are given in the following table:

		Player 2: A	Player 2: D
Player 1	A	5, 5	0, 6
	D	6, 0	1, 1

Consider an altruistic variation of the game where each player not only cares about their own payoff, but also the other player's payoff. In particular, each player's modified payoff becomes his original payoff plus α times the original payoff of the other player. For example, player 1's modified payoff to action profile (A, A) is $5 + 5\alpha$ and payoff to action profile (A, D) is $0 + 6\alpha$.

1. Write down the strategic form of this game for $\alpha = 1$. Is this game a Neighboring Kingdoms' Dilemma game (in terms of the conclusions about the outcome)? Explain your answer.

2. Find the range of values of α or which the resulting game is the Neighboring Kingdoms' Dilemma. For values of α for which the game is not the Neighboring Kingdoms' Dilemma, find its Nash equilibria.

Que 19. [Moderate] Consider the following two-player zero-sum game.

		Player II L	Player II C	Player II R
Player I	*T*	3	−3	0
	M	2	6	4
	B	2	5	6

1. Find a mixed strategy of Player I that guarantees him the same payoff against any pure strategy of Player II.
2. Find a mixed strategy of Player II that guarantees him the same payoff against any pure strategy of Player I.
3. Prove that the two strategies you found in (1) and (2) are the optimal strategies of the two players.

Que 20. [Moderate]

	Swerve	Don't Swerve
Swerve	0,0	-1,1
Don't Swerve	T,-1	-2,-2

- Find all of the pure strategy Nash equilibrium strategy profiles for this game if T $>$ 0.
- Find all of the pure strategy Nash equilibrium profiles for this game if T $<$ 0.
- If T $>$ 0, there is a mixed strategy Nash equilibrium strategy profile that is not a pure strategy Nash equilibrium. Find it and find the payoffs to each player in this equilibrium.
- In a mixed strategy Nash equilibrium with T $=$ 2, which player is more likely to swerve? If T $=$ 2, which player gets the higher expected payoff in equilibrium? Which player's equilibrium mixed strategy depends on T.
- (extra credit) Is there anything paradoxical about the results in Parts B and C? If so, what?

Que 21. [Hard] A Nash equilibrium $s^\star$ is termed strict if every deviation undertaken by a player yields a definite loss for that player, i.e. $u_i(s^\star) > u_i(s_i, s^\star_{-i})$ for each player $i \in N$and each strategy $s_i \in S_i \backslash \{s_i\}$.
(a) Prove that if the process of iterative elimination of strictly dominated strategies results in a unique strategy vector $s^\star$, then $s^\star$ is a strict Nash equilibrium, and it is the only Nash equilibrium of the game.
(b) Prove that if $s^\star = (s_i^\star)$ for $i = 1$ to n is a strict Nash equilibrium, then none of the strategies $s_i^\star$ can be eliminated by iterative elimination of dominated strategies (under either strict or weak domination)

Que 22. [Hard] Prove that the result of iterated elimination of strictly dominated strategies (that is, the set of strategies remaining after the elimination process has been completed) is independent of the order of elimination. Deduce that if the result of the elimination process is a single vector $s^\star$, then that same vector will be obtained under every possible order of the elimination of strictly dominated strategies.

Que 23. [Hard] Prove directly that a strictly dominated strategy cannot be an element of a game's equilibrium. In other words, show that in every strategy vector in which there is a player using a strictly dominated strategy, that player can deviate and increase his payoff.

Que 24. [Hard] In a first-price auction, each buyer submits his bid in a sealed envelope. The winner of the auction is the buyer who submits the highest bid, and the amount he pays is equal to what he bid. If several buyers have submitted bids equal to the highest bid, a fair lottery is conducted among them to choose one winner, who then pays his bid.
(a) In this situation, does the strategy $\beta_i^\star$ of buyer i, in which he bids his private value for the item, weakly dominate all his other strategies? Justify your answer.
(b) Find a strategy of buyer i that weakly dominates strategy $\beta_i^\star$.

Que 25. [Hard] Prove if player i has a (weakly) dominant strategy, then it is his (not necessarily unique) maxmin strategy. Moreover, this strategy is his best reply to every strategy vector of the other players.

Que 26. [Hard] Prove if in a game in which every player i has a strategy $s_i^\star$ that strictly dominates all of his other strategies, then the strategy vector $(s_1^\star, ..., s_n^\star)$ is the unique equilibrium point of the game as well as the unique vector of maxmin strategies.

Que 27. [Hard] Consider the game represented in the table below, where Player 1 chooses the row and Player 2 chooses the column.

	Turn	Don't Turn
Turn	0,0	-1,1
Don't Turn	T,-1	-2,-2

1. Find all of the pure strategy Nash equilibrium strategy profiles for this game if $T > 0$
2. Find all of the pure strategy Nash equilibrium profiles for this game if $T < 0$.
3. If $T > 0$, there is a mixed strategy Nash equilibrium strategy profile that is not a pure strategy Nash equilibrium. Find it and find the payoffs to each player in this equilibrium.
4. In a mixed strategy Nash equilibrium with $T = 2$, which player is more likely to turn? If $T = 2$, which player gets the higher expected payoff in equilibrium? Which player's equilibrium mixed strategy depends on T.
5. Is there anything paradoxical about the results in Parts 3 and 4? If so, what?

Que 28. [Hard] An embezzler wants to hide some stolen money. An inspector is looking for the stolen money. There are two places that the embezzler can put the money. One place is difficult to access and one is easy to access. The inspector only has time to look in one of the two places. It is more costly to hide the money in the difficult place than in the easy place and also more costly for the inspector to look in the difficult place than in the easy case. The payoffs are as follows.

- If the embezzler hides the money in the difficult place and the inspector looks in the difficult place, the payoff is 0 for the embezzler and 2 for the inspector.
- If the embezzler hides the money in the difficult place and the inspector looks in the easy place, the payoff is 2 for the embezzler and 1 for the inspector.
- If the embezzler hides the money in the easy place and the inspector looks in the difficult place, the payoff is 3 for the embezzler and 0 for the inspector.
- If the embezzler hides the money in the easy place and the inspector looks in the easy place, the payoff is 1 for the embezzler and 3 for the inspector.

1. Write the above game in normal form.
2. Find a Nash equilibrium in mixed strategies for this game.
3. What would be the best move for embezzler? Justify.
4. What would be the best move for inspector? Justify.

Que 29. [Hard] Consider the following bargaining problem: $20 dollars needs to be split between Jack and Jill. Jill gets to make an initial offer. Jack then gets to respond by either accepting Jill's initial offer or offering a counter offer. Finally, Jill can respond by either accepting Jakes offer or making a final offer. If Jake does not accept Jill's final offer both Jack and Jill get nothing. Jack discounts the future at 10% (i.e. future earnings are with 10% less than current earnings) while Jill discounts the future at 20%. Calculate the Nash equilibrium of this bargaining problem.

Que 30. [Hard] Hawk-Dove.

1. Two animals are fighting over some resources which gives them benefit v. Each can be a dove (passive) or a hawk (aggressive). Each prefers to be aggressive if its opponent is passive and passive if its opponent is aggressive. Given its own stance, it prefers the outcome in which its opponent is passive to that in which its opponent is aggressive. If both are aggressive, both animals incur cost c. Write down the payoff matrix and find all pure Nash equilibria.
2. Hawk-Dove-Bourgeois
   Suppose a new strategy becomes available to the players of the Hawk-Dove dove game. We will call the strategy "Bourgeois". The Bourgeois strategy says that the player who arrived first will play like a Hawk, while the player who arrived second will play like

a Dove. When two players are matched, it is randomly determined who arrived first between them. Write down the payoff matrix for this modification of the Hawk-Dove game and find all pure Nash equilibria. Do they differ from the Nash equilibria of the Hawk-Dove game without Bourgeois?

Que 31. [Hard] Prove that in any two-player game,

$$\max_{\sigma_I \in \Sigma_I} \min_{\sigma_{II} \in \Sigma_{II}} u_I(\sigma_I, \sigma_{II}) = \max_{\sigma_I \in \Sigma_I} \min_{s_{II} \in S_{II}} u_I(\sigma_I, s_{II}).$$

That is, given a mixed strategy of Player I, Player II can guarantee that Player I will receive the minimal possible payoff by playing a pure strategy, without needing to resort to a mixed strategy.

Que 32. [Hard] A two-player game is symmetric if the two players have the same strategy set $S_1 = S_2$ and the payoff functions satisfy $u_1(s_1, s_2) = u_2(s_2, s_1)$ for each $s_1, s_2 \in S_1$. Prove that the set of equilibria of a two-player symmetric game is a symmetric set: if (s_1, s_2) is an equilibrium, then (s_2, s_1) is also an equilibrium.

Que 33. [Hard] Consider the two-player game in the figure below, in which each player has three pure strategies.

		Player II		
		L	*C*	*R*
	T	3	–3	0
Player I	*M*	2	6	4
	B	2	5	6

1. Prove that $([1/3(T), 1/3(M), 1/3(B)];\ [1/3(L), 1/3(C), 1/3(R)])$ is the game's unique equilibrium.
2. Check that if Player I deviates to T , then Player II has a reply that leads both players to a higher payoff, relative to the equilibrium payoff. Why, then, will Player II not play that strategy?