CS711: Introduction to Game Theory and Mechanism Design

Midsem – Semester 1, 2018-19, Computer Science and Engineering,
Indian Institute of Technology Kanpur
Total Points: 45, Time: 2 hours, ATTEMPT ALL QUESTIONS

1. Let s_i be a strictly dominated strategy of player i in a normal form game. Is there a correlated equilibrium under which s_i is chosen with positive probability, i.e., $\sum_{s_{-i} \in S_{-i}} \pi(s_i, s_{-i}) > 0$. Justify your answer. [This means if your answer is affirmative, then give an example, if not, prove why.] **5 points.**

   Solution: Suppose there is a correlated equilibrium under which s_i is chosen with positive probability, i.e., $\sum_{s_{-i} \in S_{-i}} \pi(s_i, s_{-i}) > 0$ for some s_{-i} and π is a correlated equilibrium.
   By the definition of the correlated equilibrium

$$\sum_{s_{-i} \in S_{-i}} \pi(s_i, s_{-i}) u_i(s_i, s_{-i}) \geq \sum_{s_{-i} \in S_{-i}} \pi(s_i, s_{-i}) u_i(t_i, s_{-i}) \ , \forall t_i \tag{1}$$

   which implies

$$\sum_{s_{-i} \in S_{-i}} \pi(s_i, s_{-i}) \left[u_i(s_i, s_{-i}) - u_i(t_i, s_{-i}) \right] \geq 0 \ , \forall t_i \tag{2}$$

   since s_i is strictly dominated, $\exists \bar{t}_i$ such that $u_i(\bar{t}_i, s_{-i}) > u_i(s_i, s_{-i}), \forall s_{-i}$. Hence the inequality in equation 2 cannot hold for $\bar{t}_i$,which is a contradiction.

   Hence, There cannot be a correlated equilibrium under which s_i will be chosen with positive probability ■

2. In the lines of the definitions of implementability of social choice functions used in the class, we extend this to **implementation in correlated strategies**. We say that an SCF f is implemented in correlated equilibrium by a mechanism $\langle (M_1, \dots, M_n), g \rangle$

   (a) if there exists a correlated strategy π, with $\sum_{\theta \in \Theta} \pi(\theta) = 1, \pi(\theta) \geqslant 0, \forall \theta \in \Theta$, and a strategy profile $(s_1, \dots, s_n)$, with $s_i : \Theta_i \mapsto M_i, \forall i \in N$ s.t.

$$\mathbb{E}_{\pi(\theta_i, \theta_{-i})} u_i(g(s_i(\theta_i), s_{-i}(\theta_{-i})), \theta_i) \geqslant \mathbb{E}_{\pi(\theta_i, \theta_{-i})} u_i(g(m_i', s_{-i}(\theta_{-i})), \theta_i), \forall m_i' \in M_i,$$

   (b) if for all $\theta \in \Theta$

$$g(s_i(\theta_i), s_{-i}(\theta_{-i})) = f(\theta).$$

   Prove that revelation principle holds here as well. That is, if f is implementable in correlated equilibrium, then f is **correlated incentive compatible**, i.e., the direct mechanism with $M_i = \Theta_i$ and $g \equiv f$ is also implementable in correlated equilibrium. Show all steps and explain the arguments clearly. **5 points.**

   Solution:[Sketch] Replace m_i' with $s_i(\theta_i')$ and use the implementability argument similar to the previous proofs done in the class. ■

3. There are $n \geqslant 1$ partners who together own a firm. Each partner i chooses an effort level $x_i \geqslant 0$, resulting in total profit $g(y)$ for their firm, where y is the sum of all partners' efforts. The profit function $g : \mathbb{R}_{\geqslant 0} \mapsto \mathbb{R}_{\geqslant 0}$ satisfies $g(0) = 0$ and it is twice differentiable with $g' > 0$, and $g'' \leqslant 0$. The profit is shared equally by the partners, and each partner's effort gives him or her a quadratic cost. The resulting utility for each partner i is therefore

$$u_i(x_1, \dots, x_n) = \frac{1}{n} g(x_1 + \dots + x_n) - \frac{1}{2} x_i^2.$$

Each partner i has to decide his or her effort x_i without observing the others' efforts.

(a) Show that the game has exactly one Nash equilibrium (in pure strategies), and show that all partners make the same effort, x^*, in equilibrium. (A precise and formal argumentation is required.) Is the individual equilibrium effort x^* increasing or decreasing in n, or is it independent of n? Is the aggregate equilibrium effort, $y^* = nx^*$, increasing or decreasing in n, or is it independent of n? **4 + 2 + 2 points.**

(b) Suppose that the partners can pre-commit to a common effort level, the same for all. Let $\hat{x}$ be the common effort level that maximizes the sum of the partners' utilities. Characterize $\hat{x}$ in terms of an equation, and compare this level with the equilibrium effort x^* in (a), for $n = 1, 2, \dots$. Are the partners better off now than in the equilibrium in (a)? How does this depend on n? Explain! **2 + 1 + 2 points.**

(c) Solve (a) and (b) explicitly and find the values of x^* and $\hat{x}$ in the special case when g is linear, i.e., $g(y) = y$. **1 + 1 points.**

Solution:

(a) The first order condition of optimality for partner i is

$$\frac{\partial u_i}{\partial x_i} = \frac{1}{n} g' \left(\sum_{j \in N} x_j \right) - x_i$$

And the second order condition is

$$\frac{\partial^2 u_i}{\partial x_i^2} = \frac{1}{n} g'' \left(\sum_{j \in N} x_j \right) - 1 \qquad < 0$$

u_i is strictly concave and is maximized **uniquely** at

$$x_i = \frac{1}{n} g' \left(\sum_{j \in N} x_j \right) > 0 \qquad \forall i \in N$$

RHS in the above equation is same for all, hence $x_i = x^* \ \forall i \in N$ and,

$$x^* = \frac{1}{n} g' (nx^*)$$

$$nx^* = g' (nx^*)$$

[Alternative argument for uniqueness: This is the equation every equilibria must satisfy, but we see the g' is continuous and non-increasing ($g'' \leq 0$) and nx^* is increasing. Hence, the equation can have only one solution. Therefore, x^* is unique.]

If we write the same equation in terms of aggregate effort $y^* = nx^*$ and the equation is $y^* = g'(y^*)$, and the solution is insensitive to n. Hence, the aggregate effort is independent of n and $x^* = \frac{y^*}{n}$ is decreasing in n.

(b) For pre-commited effort z, the expression of sum of utilities is

$$W(z) = g'(nz) - \frac{nz^2}{2}$$

By definition $\hat{x}$ maximizes $W(\cdot)$

$$W(\hat{x}) \geq W(z) \qquad \forall z \tag{3}$$

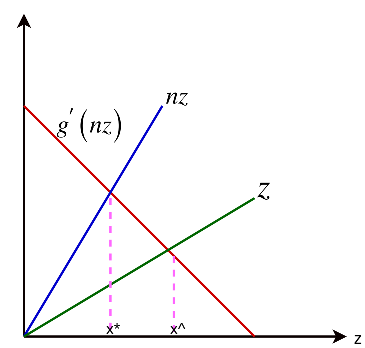
$$W'(z) = ng'(nz) - nz$$

$$W''(z) = n^2 g''(nz) - n \qquad < 0$$

Hence, $ng'(n\hat{x}) = n\hat{x}$

$$\Rightarrow g'(n\hat{x}) = \hat{x} \tag{4}$$

By a similar logic as before, $\hat{x}$ is also unique.



A representative figure 3b (since g' is a non-increasing function) explains why $x^* \leq \hat{x}$, equality holds only when $n = 1$.

Also, the utility at x^* for every player is

$$\frac{1}{n} g(nx^*) - \frac{1}{2} x^{*\,2} \leq \frac{1}{n} g(n\hat{x}) - \frac{1}{2}\hat{x}^2 \qquad \text{by (3)}$$

Hence, individual effort and individual utility both are smaller in Nash equilibrium.

Also, note that for a given x, $g'(nx)$ does not increase as n increases (as $g'' \leq 0$). From the figure 3b, it is evident that the downward slope of $g'(nz)$ will increase if n increases, and the intersection with the z line will be weakly earlier. Hence $\hat{x}$ weakly decreases with n.

(c) $g\left(y\right) = y \Rightarrow x^* = \frac{1}{n}$, $\hat{x} = 1$

■

4. Café Coffee Day and Domino's Pizza are contemplating providing stalls during Antaragni 2018 on the adjoining street in front of the hostel blocks. Assume that the locations can be modeled as an interval $[0,1]$. Also assume that the student population is uniform over this interval, and every student prefers to go to the stall that is closest to him/her – in case of a tie, the stalls get equal proportion of the population. Assume, for simplicity, that the proportion can be any real number in $[0,1]$. The stalls' utilities are the fraction of the population they get in their stall.

   (a) Describe this situation formally as a two player normal form game. Explain fully. [This entails the description of the players, strategies, utilities that are needed for the game.]

   3 points.

   (b) Find a pure strategy Nash equilibrium of this game. Is the PSNE unique?

   3 + 1 points.

   (c) Suppose, now Pizza Hut decides to enter this game and provide a stall. How will the PSNE change? The rules remain the same – every student prefers the closest stall, and in case of ties, the stalls split the population equally. **3 points.**

Solution:

(a) Say 1 is the left player and 2 is the right player. Left player's position is l and right player's is r, $l \leq r$.

$$u_1(l,r) = \begin{cases} \frac{l+r}{2} & l < r \\ \frac{1}{2} & l = r \end{cases}$$

$$u_2(l,r) = \begin{cases} 1 - \frac{l+r}{2} & l < r \\ \frac{1}{2} & l = r \end{cases}$$

(b) If $l < r$ then each player has an incentive to move closer to the other one which strictly increases her payoff. Hence, none of these cases can yield a PSNE. At $l = r \neq \frac{1}{2}$, both agents gain by moving slightly towards the center – this rules out those cases for PSNE as well. For $l = r = \frac{1}{2}$, no agent has any benefit from moving – the utilities are maximum equal to $\frac{1}{2}$. This is PSNE and by the arguments above, the PSNE is unique.

(c) Consider the position of the third player as x, without loss of generality (the labels of the players do not matter) $l < x < r$. Clearly, this is not a PSNE since any player l or r improves by moving right or left. If $l = x < r$, still it is beneficial for r to move left and similaraly $l < x = r$ cannot be a PSNE. If all the overlap, then each of them will get $\frac{1}{3}$, but any one of them can move to larger side and get atleast close to $\frac{1}{2}$. So, no PSNE exists.

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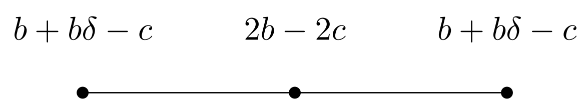
5. **Strategic network formation game**: consider a group of individuals who are considering forming a network. A link or edge in this network is formed on a **mutual agreement** of both the individuals, but can be severed **unilaterally**. If a link exists between i and j, then both get a benefit of b from this connection. But maintaining this link is expensive, therefore the cost c is incurred by both individuals. If k has a link with j but not with i, then i gets an indirect benefit from this connection given by $b\delta$, where $1 > \delta > 0$, but does not incur any cost for this *indirect* benefit. Similarly for any two nodes that are at a distance d hops away (only a shortest path is counted) get a benefit of $b\delta^{d-1}$ each. The utilities of the nodes in an example network is given in the following figure.



The cost is incurred only for direct connections. Every node in this network is strategic, i.e., each of them maintains or severs a link based on whichever gives them a higher utility. Consider the following cases and explain in detail what kind of networks are expected to form in each case. To be consistent, assume that the nodes are numbered $1, \ldots, n$, and a node moves after all its previous nodes have stabilized with their moves.

(i) $b < c$ **3 points.**

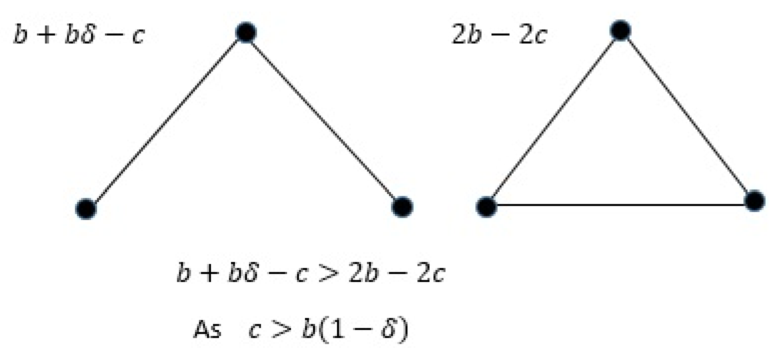
(ii) $b(1 - \delta) < c \leqslant b$ **4 points.**

(iii) $c \leqslant b(1 - \delta)$ **3 points.**

Solution:[Sketch]

(i) $b < c$, in this scenario, no link can form, since to build a link, the net benefit is $b - c < 0$ and by not keeping a link, it is 0. Since links form on mutual agreement and deletes unilaterally. Thus a **EMPTY** network will be formed.

(ii) $b \geq c > b(1 - \delta)$. In this case, creating one link is beneficial than creating no link but if a vertex is indirectly connected to another vertex, then it is not more beneficial to directly connect to that link.



Hence one node will connect to all others and **STAR** network will be formed.

(iii) $c \leq b(1-\delta)$, here creating direct links are always more beneficial than any indirect connection. Hence **COMPLETE** network will form.

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That's all folks!