# CS711: Introduction to Game Theory and Mechanism Design

**Endsem – Semester 1, 2018-19, Computer Science and Engineering,**
**Indian Institute of Technology Kanpur**
**Total Points: 40, Time: 2 hours, ATTEMPT ALL QUESTIONS**

1. Let $A$ be a finite set of alternatives and $f : \mathcal{P}^n \to A$ be a social choice function that is unanimous and strategy-proof, $n \geqslant 3$. Suppose $|A| \geqslant 3$, and $\mathcal{P}$ denotes the set of all possible strict preferences over $A$.

   Now, consider another social choice function $g : \mathcal{P}^2 \to A$ defined as follows. The SCF $g$ only considers profiles of two agents, denote these two agents as 1 and 2. For every $(P_1, P_2) \in \mathcal{P}^2$, let
   $$g(P_1, P_2) = f(P_1, P_2, P_1, P_1, \ldots, P_1),$$
   i.e., the outcome of $g$ at $(P_1, P_2)$ coincides with the outcome of $f$ at the profile where agents 1 and 2 have types $P_1$ and $P_2$ respectively, and all other agents have type $P_1$.

   Show that $g$ is a dictatorship SCF.

   [Hint: you may use the Gibbard-Satterthwaite characterization result.] **10 points.**

   **Solution:** Since $f$ is unanimous, it is clear that $g$ will be unanimous. Since $f$ is unanimous and strategy-proof and $|A| \geqslant 3$, by the GS Theorem, $f$ is a dictatorship. We consider three possible cases.

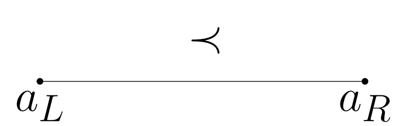
   Case 1. Agent 1 is the dictator of $f$. This means agent 1 gets his top alternative in $f$ and also in $g$. We claim that agent 1 is also the dictator of $g$. Agent 1 cannot manipulate $g$ since he gets his top choice. Since agent 1 is the dictator of $f$, for any $P_1$ and $P_2, P_2'$, we have $g(P_1, P_2) = g(P_1, P_2') = f(P_1, P_2, \underbrace{P_1, \ldots, P_1}_{(n-2) \text{ times}}) = f(P_1, P_2', \underbrace{P_1, \ldots, P_1}_{(n-2) \text{ times}}) = P_1(1)$. So, clearly agent 2 cannot manipulate $g$, and $g$ is a dictatorship.

   Case 2. Agent 2 is the dictator of $f$. This means agent 2 gets his top alternative in $f$ and also in $g$. In this case, a very similar argument as case 1 shows that agent 2 is also the dictator of $g$.

   Case 3. Agent $i \notin \{1, 2\}$ is the dictator of $f$. Since $g$ is defined when all preferences of agents except 1 and 2 are identical to the preference of agent 1, if the dictator is an agent except the first two agents, $g$ will always pick the top alternative of player 1. I.e., $g(P_1, P_2) = f(P_1, P_2, \underbrace{P_1, \ldots, P_1}_{(n-2) \text{ times}}) = P_1(1)$. Hence, agent 1 is also the dictator of $g$.

   This concludes the proof that $g$ is a dictatorship. ■

2. Let $A$ be a finite set of alternatives and $\prec$ be an intrinsic linear order over $A$ – see the figure below for an illustration.



Suppose $a_L, a_R \in A$ be two alternatives such that $a_L \prec a$ for all $a \in A \setminus \{a_L\}$ and $a \prec a_R$ for all $a \in A \setminus \{a_R\}$ - in other words, $a_L$ is the "left-most" alternative and $a_R$ is the "right-most" alternative with respect to $\prec$.

Let $\mathcal{S}$ be the set of all possible single-peaked strict orderings over $A$ with respect to $\prec$. An SCF $f : \mathcal{S}^n \to A$ maps the set of preference profiles of $n$ agents to $A$.

Let $P_i(1)$ denote the peak of agent $i$ in $P_i$. Suppose $f$ satisfies the following property (call it property $\Pi$). There is an alternative $a^* \in A$ such that for any preference profile $(P_1, \ldots, P_n) \in \mathcal{S}^n$, where $P_i(1) \in \{a_L, a_R\}$ for all $i \in N$ with at least one agent's peak at $a_L$ and at least one agent's peak at $a_R$, $f(P_1, \ldots, P_n) = a^*$.

Suppose $f$ is strategy-proof, Pareto efficient, anonymous, and satisfies property $\Pi$. Then, give a precise (simplified) description of $f$ (using $a^*$), i.e., for every preference profile $P$, what is $f(P)$?

[Hint: you may use the Moulin characterization result and argue where the phantom peaks must be located.] **10 points.**

**Solution:** Since $f$ is strategy-proof, efficient, and anonymous, it must be a median voter SCF. Further, for any preference ordering $(P_1, \ldots, P_n) \in \mathcal{S}^n$, where $P_i(1) \in \{a_L, a_R\}$ for all $i \in N$, $f(P_1, \ldots, P_n) = a^*$. This means that all the phantom peaks are located at $a^*$. This follows from the fact that the location of a phantom peak can be obtained as outcome of $f$ by placing suitable number of agent peaks at $a_L$ and the remaining agent peaks at $a_R$. If all the phantom peaks were not at $a^*$, then at least in one configuration of the $P_i(1) \in \{a_L, a_R\}$ for all $i \in N$ with at least one peak at the extreme left and right positions the median would have been different than $a^*$, which leads to a contradiction.

[A solution that shows that $f$ with all phantom peaks at $a^*$ satisfies these four properties (strategy-proof, Pareto efficient, anonymous, and property $\Pi$) is not the correct solution, since it exhibits *one* such SCF. The question asks for the reverse – if those properties are satisfied, what will necessarily be the structure of $f$. The answer turns out to be the same – $f$ with all phantom peaks at $a^*$ – but it also shows that no other SCF can satisfy those four properties and that this is a necessary condition.]

Hence, the outcome of $f$ is median of agent peaks and $(n-1)$ phantom peaks located at $a^*$. This simplifies $f$ as follows. For any preference profile $P \equiv (P_1, \ldots, P_n)$, if $P_i(1) \succ a^*$ for all $i \in N$, then $f(P)$ is the "left-most" agent peak, i.e., $f(P) = P_j(1)$, where $P_j(1) = P_k(1)$ or $P_k(1) \prec P_j(1)$ for all $k \in N \setminus \{j\}$. Similarly, if $a^* \succ P_i(1)$ for all $i \in N$, then $f(P)$ is the "right-most" agent peak, i.e., $f(P) = P_j(1)$, where $P_j(1) = P_k(1)$ or $P_j(1) \prec P_k(1)$ for all $k \in N \setminus \{j\}$. Otherwise, $f(P) = a^*$. ■

3. Three friends, numbered 1, 2, and 3, are planning to choose a movie to watch together. The candidate movies are A, B, and C (exactly one of them can be chosen). The values (derived from their film interests) that the friends get from watching these movies are given in the table below.

| | A | B | C |
|---|---|---|---|
| 1 | 7 | 5 | 1 |
| 2 | 7 | 5 | 2 |
| 3 | 2 | 5 | 10 |

They agree that the decision should be 'efficient' as much as possible, i.e., it should maximize the sum value of all the agents, but they are also concerned whether (a) this can be done truthfully, and (b) using only internal transfers, i.e., keeping it budget balanced.

(i) If they decide to ask the agents their valuations and decide the efficient choice of movie without any monetary transfers, will the properties of (a) and (b) be satisfied? Explain why or why not. **2 points.**

**Solution:** Since there is no transfer, (b) is certainly satisfied. But, this mechanism does not satisfy (a), since the agents can overbid the valuations of their most preferred alternative, and underbid the less preferred one and that strategy weakly dominates truthtelling. ■

(ii) If monetary transfer is allowed and that results in a quasi-linear utility to the agents (the friends), can efficient choice of movie be made in a truthful manner? Provide a payment rule that implements the efficient choice. What is the movie decided and how much should each agent pay? What is the net utility of each of the players? Is this mechanism budget balanced? **1 + 2 + 3 + 3 + 1 points.**

**Solution:** Yes, the efficient choice can be implemented in a truthful manner using VCG payment. The payment for agent $i$ is $p_i(v) = \sum_{j \neq i} v_j(a^*_{-i}(v)) - \sum_{j \neq i} v_j(a^*(v))$. Calculating this for different players, we get

$$
\begin{aligned}
p_1 &= 12 - 9 = 3 & \qquad u_1 &= 7 - 3 = 4 \\
p_2 &= 11 - 9 = 2 & \qquad u_2 &= 7 - 2 = 5 \\
p_3 &= 14 - 14 = 0 & \qquad u_3 &= 2 - 0 = 2
\end{aligned}
$$

The mechanism is not budget balanced since players 1 and 2 make a positive payment, and player 3 does not make or consume any payment. ■

(iii) Suppose they decide on a different mechanism instead. Agent 2's values are not considered in the movie decision process. The mechanism chosen in item (ii) above is used for agents 1 and 3, the monetary surplus generated is transferred to agent 2. Does this mechanism satisfy properties (a) and (b)? Clearly explain why. What are the net utilities of the players? Is this mechanism efficient?

**3 + 3 + 2 points.**

**Solution:** Without agent 2, the outcome of the same mechanism will be $C$ which is not the efficient one. But this mechanism satisfies both (a) and (b). The mechanism is budget balanced by construction. It is truthful since, after removal of player 2, players 1 and 3 are involved in a VCG mechanism only between themselves, and this mechanism is known to be truthful. Player 2 cannot change its payment or decision of the movie in any way, therefore the mechanism is trivially truthful for player 2 as well. The payments and utilities are as follows.

$$
\begin{aligned}
p_1 &= 10 - 10 = 0 & \qquad u_1 &= 1 - 0 = 1 \\
p_3 &= 7 - 1 = 6 & \qquad u_3 &= 10 - 6 = 4 \\
p_2 &= -6 & \qquad u_2 &= 2 - (-6) = 8
\end{aligned}
$$

Clearly, the mechanism is not efficient as it chooses the outcome C while the efficient one is A. ■