CS711: Introduction to Game Theory and Mechanism Design

Assignment 1 – Semester 1, 2018-19
Computer Science and Engineering
Indian Institute of Technology Kanpur
Total Points: 30, Time: 2 days, ATTEMPT ALL QUESTIONS

A general information: here are the values of certain variables that you will need to use in the solutions. **Write on top of your answerscript only the values of the variables x and y, and nothing more.** If the last two digits of your roll number are a and b respectively, i.e., 67 implies $a = 6, b = 7$, then, for the following questions:

$$x = \begin{cases} a & \text{if } a > 6 \\ a + 18 & \text{if } a \leqslant 6 \end{cases}$$

$$y = \begin{cases} b & \text{if } b > 9 \\ b + 9 & \text{if } b \leqslant 9 \end{cases}$$

1. Find all Nash Equilibria of this game (includes pure and mixed).

		Column Player	
		A	B
Row Player	A	$4, 0$	$2, 4$
	B	$0, x + y + 1$	$3 + x, y$

Clearly state what the equilibrium strategies of the two players are. **5 points.**

Solution:[sketch] Clearly, this game does not admit any pure strategy Nash equilibrium, which can be checked by individually checking the four strategy profiles. Since there are only two strategies per player, it is clear that both the strategies must be in the support of the MSNE. Let the probabilities with which row and column players choose A are p and q respectively. Use the characterization theorem of MSNE to find the values of p and q. For example, equating the expected utility for the strategies in the support of player 2, we get

$$(x + y + 1)(1 - p) = 4p + y(1 - p)$$

$$\implies \frac{p}{1 - p} = \frac{x + 1}{4}$$

$$\implies p = \frac{x + 1}{x + 5}$$

Similarly, one can find, $q = \frac{x+1}{x+5}$. ■

2. Consider the market share contest between two largest cloud service providers, Microsoft and Amazon. The price of every unit of cloud resource in the market with Microsoft (Firm 1) and Amazon (Firm 2) as the providers is

$$P(q_1, q_2) = (x + y) - q_1 - q_2$$

where q_1 and q_2 is the amount of the resource produced by the two firms respectively. The utility of Firm 1 is

$$u_1(q_1, q_2) = P(q_1, q_2) \cdot q_1 - q_1$$

and for Firm 2 is

$$u_2(q_1, q_2) = P(q_1, q_2) \cdot q_2 - \frac{x + y}{2} \cdot q_2$$

(a) Firms move simultaneously.

(i) How much should each firm produce? **3 points.**

(ii) What is each firm's utility? Show all steps for both the answers. **3 points.**

(b) Suppose in the same setting, Firm 1 moves first and decides its production of resource q_1, and then Firm 2 watches q_1 and decides how much resource q_2 to produce. The price in the market and the individual utilities have the same expressions as before.

(i) How much should each firm produce in this changed scenario? **5 points.**

(ii) What is each firm's utility? Show all steps for both the answers. **4 points.**

Solution:[sketch]

(a) **Simultaneous Move:**

(i) Firm 1's problem is

$$\max_{q_1} \quad (x + y - q_1 - q_2)q_1 - q_1$$

and Firm 2's problem is

$$\max_{q_2} \quad (x + y - q_1 - q_2)q_2 - \frac{x + y}{2} q_1$$

The first order conditions of optimality are respectively

$$x + y - 2q_1 - q_2 - 1 = 0 \quad (1)$$

$$x + y - 2q_2 - q_1 - \frac{x + y}{2} = 0 \quad (2)$$

Solving simultaneously gives us

$$q_1 = \frac{x + y}{2} - \frac{2}{3}$$

$$q_2 = \frac{1}{3}$$

Hence the quantities produced at the PSNE of this simultaneous move game is $q_1^* = \frac{x+y}{2} - \frac{2}{3}$ and $q_2^* = \frac{1}{3}$.

(ii) Price at this equilibrium is given by

$$\left(x + y - \frac{x+y}{2} + \frac{2}{3} - \frac{1}{3}\right) = \frac{x+y}{2} + \frac{1}{3}.$$

Using the utility expressions, we get

$$\begin{aligned} u_1(q_1^*, q_2^*) &= \left(\frac{x+y}{2} - \frac{2}{3}\right)\left(\frac{(x+y)}{2} + \frac{1}{3} - 1\right) \\ &= \left(\frac{x+y}{2} - \frac{2}{3}\right)^2, \quad \text{and} \\ u_2(q_1^*, q_2^*) &= \frac{1}{3}\left(\frac{x+y}{2} + \frac{1}{3} - \frac{x+y}{2}\right) \\ &= \frac{1}{9}. \end{aligned}$$

(b) **Sequential Move:**

(i) We solve the problem via backward induction by first considering Firm 2's problem given Firm 1's choice q_1

$$\max_{q_2} \quad (x + y - q_1 - q_2)q_2 - \frac{x+y}{2}q_2$$

The first order condition of optimality for Firm 2 is as before (expression 2)

$$\begin{aligned} x + y - 2q_2 - q_1 - \frac{x+y}{2} &= 0 \\ \implies q_2 &= \frac{x+y}{4} - \frac{q_1}{2} \end{aligned}$$

And the best response is

$$q_2 = \begin{cases} \frac{x+y}{4} - \frac{q_1}{2} & \text{if } q_1 \leqslant \frac{x+y}{2} \\ 0 & \text{otherwise} \end{cases}$$

We consider the following two cases.
Case 1: Suppose $q_1 \leqslant \frac{x+y}{2}$. Then solving backwards, Firm 1's problem now becomes

$$\max_{q_1} \quad \left(x + y - q_1 - \frac{x+y}{4} + \frac{q_1}{2}\right) q_1 - q_1$$

which has the slope $\frac{3}{4}(x+y) - q_1 - 1$ that is nonnegative for all $q_1 \leqslant \frac{x+y}{2}$ (for all the choices of x and y), which means Firm 1 should want to produce more at all points in that range. Hence the maxima in this region lies at $q_1 = \frac{x+y}{2}$.
Case 2: Suppose $q_1 > \frac{x+y}{2}$, where $q_2 = 0$ is a best response by Firm 2. Firm 1's problem now becomes

$$\max_{q_1} \quad (x + y - q_1)q_1 - q_1$$

which has the slope $(x+y) - 2q_1 - 1$ that is negative for all $q_1 > \frac{x+y}{2}$. Hence the maxima in this region lies at $q_1 = \frac{x+y}{2}$.
Considering both the cases, the optimal q_1^* and q_2^* for a sequential move will be

$$\begin{aligned} q_1^* &= \frac{x+y}{2} \\ q_2^* &= 0 \end{aligned}$$

(ii) Price at this equilibrium is given by

$$\left(x + y - \frac{x+y}{2} - 0\right) = \frac{x+y}{2}.$$

Using the utility expressions, we get

$$\begin{aligned} u_1(q_1^*, q_2^*) &= \left(\frac{x+y}{2}\right)\frac{x+y}{2} - \frac{x+y}{2} \\ &= \left(\frac{x+y}{2}\right)^2 - \frac{x+y}{2}, \quad \text{and} \\ u_2(q_1^*, q_2^*) &= 0. \end{aligned}$$

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3. Three lions live in a jungle. They can hunt in the morning or evening, but not both. If a lion gets the morning time slot alone, it gets a payoff of x. If it gets the evening time slot alone, it gets a payoff of $x + y + 1$. If two lions hunt at the same time, both get zero.

   (i) Show that any strategy profile with two lions hunting in the morning and one in the evening is a pure-strategy Nash equilibrium. Show that any strategy profile with two lions hunting in the evening and one in the morning is a pure-strategy Nash equilibrium.

   (ii) Show that one mixed-strategy Nash equilibrium of the game is for a single lion to choose morning for sure, one lion to choose evening for sure, and the third lion to randomize however it likes.

   (iii) If all lions randomize in a symmetric mixed strategy equilibrium, with what probability do they hunt in the morning?

2+3+5 points.

Solution:[sketch]

(i) Presently the two lions (suppose L1 and L2) hunting in the morning both get zero, and L3 gets the highest possible payoff $(x + y + 1)$. Clearly L3 does not have a profitable deviation. If L1 or L2 deviated to hunt in the evening they would still get zero, and hence, they would not deviate either. The argument can be replicated for any combination of the lions.

(ii) Suppose the randomizing lion mixes his the strategies as follows. Hunts in the morning w.p. α and in the evening w.p. $(1-\alpha)$. The lions who play pure strategies of hunting in the morning and evening get expected payoffs of $x(1-\alpha)$ and $(x+y+1)\alpha$ respectively and hence deviation to any other strategy (pure or mixed) would result in getting an expected payoff $< x(1-\alpha)$ and $< (x+y+1)\alpha$ respectively. Thus those lions do not want to deviate to any mixture of those two pure strategies either. The lion that mixes his strategies gets zero from playing both pure strategies and hence any mixed strategy can do no better. Thus all lions are best responding.

(iii) Probability of no other lion hunting in morning is $(1-\alpha)^2$ and no other lion hunting in the evening is α^2. Therefore using the indifference property of the characterization result of MSNE we get

$$\begin{aligned} (1-\alpha)^2 x &= \alpha^2(x+y+1) \\ \iff \frac{1}{\alpha} - 1 &= \sqrt{\frac{x+y+1}{x}} \\ \iff \alpha &= \frac{1}{1+\sqrt{\frac{x+y+1}{x}}} \end{aligned}$$

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That's all folks!