CS711: Introduction to Game Theory and Mechanism Design

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Imperfect Information Extensive Form Games

Mixed strategy equivalent to behavioral strategy

Theorem

Let $\langle N, A, \mathcal{H}, \mathcal{X}, P, (u_i)_{i \in N}, (I_i)_{i \in N} \rangle$ be an IIEFG such that at every vertex of the game tree there are at least two actions. Every behavioral strategy has an equivalent mixed strategy if and only if each information set of player i intersects every path emanating from the root at most once.

Behavioral strategy equivalent of mixed strategy

- requires formalizing the forgetfulness of a player
- a player can forget
  - what moves he has made in the past (example 2)
  - whether he at all made a move in the past (example 3)
  - whether there was a chance move, what was the outcome of the chance move, which players played in the past, what and how many times those players played etc.
- we will formalize a definition that ensures none of the forgetfulness happens
- this will supersede the condition in the previous theorem

Games with perfect recall

- define the path choosing same action

Definition (Choice of same action at information set)

- Let $X = (x^0, x^1, \ldots, x^K)$ and $\widehat{X} = (x^0, \widehat{x}^1, \ldots, \widehat{x}^L)$ be two paths in the game tree.
- Let I_i^j be an information set of player i that intersects these two paths only at one vertex, given by x^k and $\widehat{x}^l$ respectively.
- These two paths *choose the same action at information set* I_i^j if
  - $k < K$ and $l < L$, and
  - the action at x^k leading to x^{k+1} is identical to the action at $\widehat{x}^l$ leading to $\widehat{x}^{l+1}$, denoted by $a_i(x^k \to x^{k+1}) = a_i(\widehat{x}^l \to \widehat{x}^{l+1})$.

- we are now ready to define **game with perfect recall**

Perfect recall

Definition (Perfect recall)

Player i has *perfect recall* if the following conditions are satisfied:

(a) Every information set of player i intersects every path from the root to a leaf at most once.

(b) Every two paths from the root that end in the same information set of player i pass through the same information sets of player i, and in the same order, and in every such information set the two paths choose the same action.
In other words, for every information set I_i^j of player i and every pair of vertices x and x' in I_i^j, if the decision vertices for player i on the path from root to x are $x_i^1, x_i^2, \ldots, x_i^L = x$ and his decision vertices on the path from root to x' are $x_i'^1, x_i'^2, \ldots, x_i'^{L'} = x'$, then

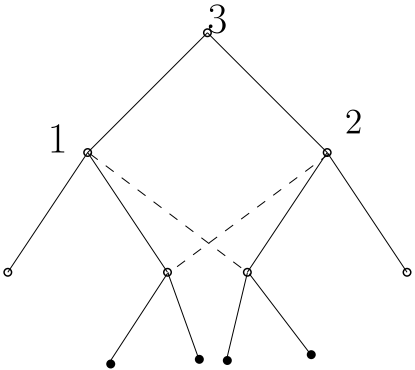
(i) $L = L'$

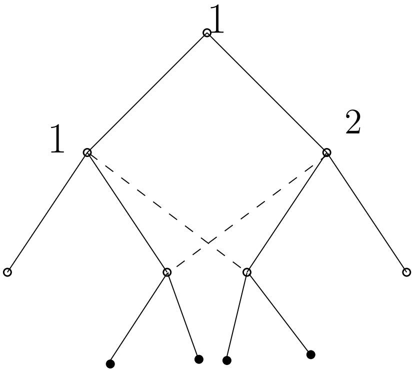
(ii) $x_i^l, x_i'^l \in I_i^k$ for some k, and

(iii) $a_i(x_i^l \to x_i^{l+1}) = a_i(x_i'^l \to x_i'^{l+1})$, for all $l = 1, \ldots, L-1$.

A game is called a game with perfect recall if every player has perfect recall.

Examples





Implications of perfect recall

- $S_i^*(x)$: set of pure strategies of player i at which he chooses the actions leading to vertex x
- in other words, intersections of members of S_i with the path from root to x
- perfect recall implies the following result

Theorem

Let i be a player with perfect recall in an IIEFG, and let x and x' be two vertices in the same information set of i. Then $S_i^*(x) = S_i^*(x')$

- Kuhn's theorem – equivalence of mixed and behavioral strategies

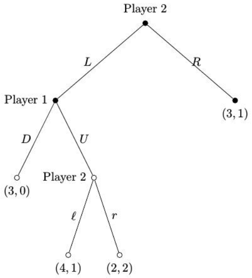
Theorem (Kuhn 1957)

In every IIEFG, if i is a player with perfect recall, then for every mixed strategy of player i there exists an equivalent behavioral strategy.

- note that perfect recall subsumes the requirement of the theorem where for every behavioral strategy there is a mixed strategy – hence this is a result that makes these two strategies equivalent

Proof outline

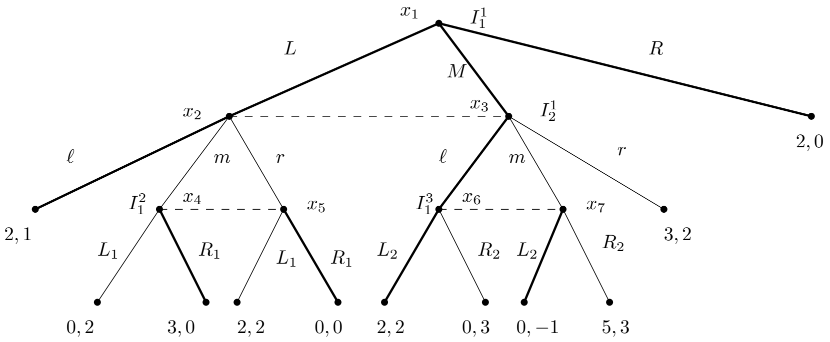
- construction from a given mixed strategy
- $\sigma_2(L\ell) = \sigma_2(Lr) = \frac{1}{3}$, $\sigma_2(R\ell) = \frac{1}{12}, \sigma_2(Lr) = \frac{1}{4}$
- need an equivalent behavioral strategy, i.e., same probabilities at every leaf node
- $\sigma_1(U) = b_1(U), \sigma_1(D) = b_1(D)$
- equations for matching probabilities
- rest of the argument shows that such a construction is possible due to the facts that the information sets cut every path at most once, and the number of times and actions with which two different paths reach two nodes in the same information set are same



Equilibrium notions in IIEFG

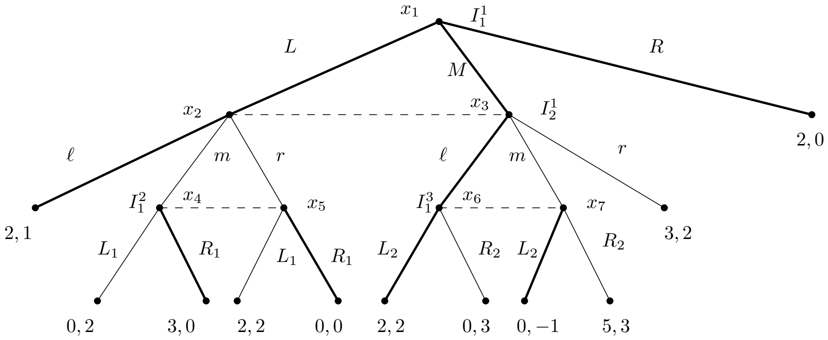
- can define subgame perfection on IIEFGs – Nash equilibrium at every subgame – includes mixed Nash too
- because of the information sets, a best response of every player cannot be defined unless
- players have a **belief** about the nodes in an information set
- example – assume games with perfect recall

Example



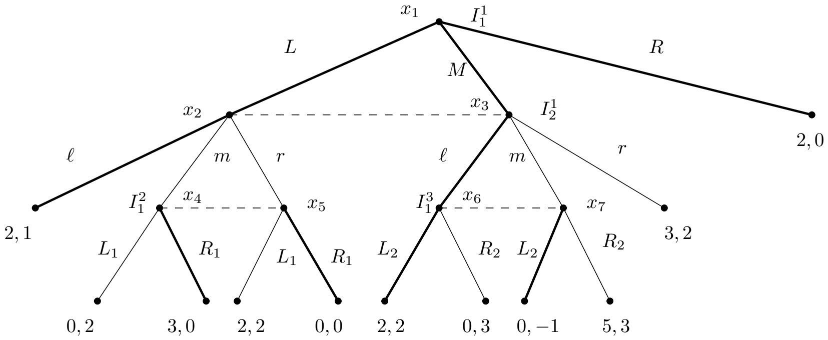
- Player 1's strategy σ_1:
  - at I_1^1, $\left[\frac{5}{12}L, \frac{4}{12}M, \frac{3}{12}R\right]$
  - at I_1^2, choose R_1
  - at I_1^3, choose L_2
- Player 2' strategy σ_2: choose ℓ
- Is this an equilibrium in a Bayesian sense? ⋆

Example (contd.)



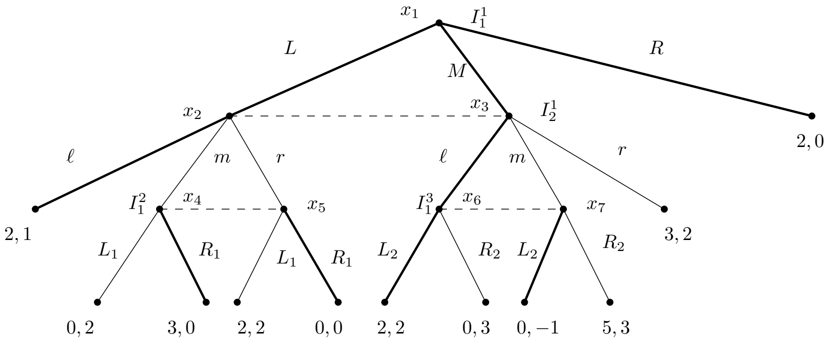
- beliefs of the players are consistent with P_σ
- **partial belief system** – beliefs defined only in some information sets
- connection between **action** and **belief** at an information set

Example (contd.)



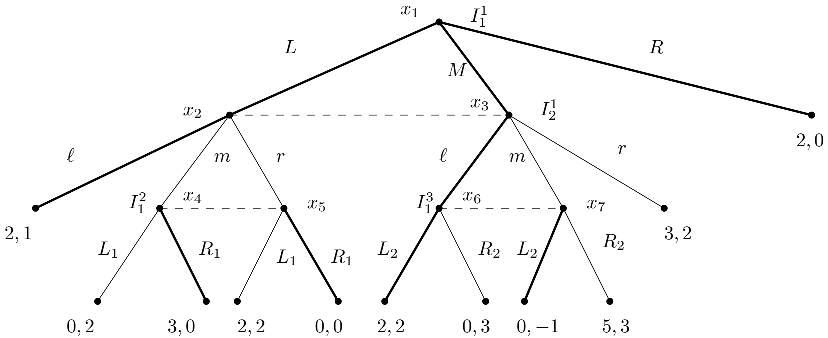
- player 1, at I_1^3, believes that x_6 is reached w.p. 1
- if the belief was $> \frac{2}{7}$ in favor of x_7
- *choose an action maximizing expected payoff at each information set* – **sequential rationality**

Example (contd.)



- player 2, at I_2^1, believes that x_3 is reached w.p. $\frac{4}{9}$
- is the action of player 2 sequentially rational with her belief?
- $\ell \rightarrow \frac{5}{9}1 + \frac{4}{9}2 = \frac{13}{9}$

Example (contd.)



- given these, player 1, at I_1^1, what will be the best response?
- in all actions, the utility is 2, but a different strategy will change the numbers for other information sets
- **strategy profile σ is sequentially rational for all the players**

Formal definitions

Definition (Belief)

Let the *information sets* of player i be $I_i = \{I_i^1, I_i^2, ..., I_i^{k(i)}\}$. In an IIEFG, the belief of player i is a map $\mu_i^j : I_i^j \to [0, 1]$, such that,

$$\sum_{x \in I_i^j} \mu_i^j(x) = 1.$$

Definition (Bayesian belief)

A belief $\mu_i := (\mu_i^j, j = 1, \dots, k(i))$ of player i is Bayesian with respect to the behavioral strategy σ, if it is derived from the strategy profile σ using Bayes' rule, i.e.,

$$\mu_i^j(x) = \frac{P_\sigma(x)}{\sum_{y \in I_i^j} P_\sigma(y)}, \forall x \in I_i^j, \ \forall j = 1, \dots, k(i).$$

Definitions (contd.)

Definition (Sequential rationality)

A strategy σ_i of player i at an information set I_i^j is **sequentially rational** given σ_{-i} and partial beliefs μ_i if $\forall \sigma_i'$

$$\sum_{x \epsilon I_i^j} \mu_i(x) u_i(\sigma_i, \sigma_{-i} | x) \geqslant \sum_{x \epsilon I_i^j} \mu_i(x) u_i(\sigma_i', \sigma_{-i} | x).$$

The pair (σ, μ) is sequentially rational if it is sequentially rational for every player, at every information set.

The pair (σ, μ) is called an **assessment**

Relationship with Nash

- sequential rationality is a **refinement** of the Nash equilibrium
- the equilibrium notion coincides with SPNE when applied to PIEFGs
- result

Theorem

In a PIEFG, a behavioral strategy profile σ is an SPNE if and only if the pair $(\sigma, \widehat{\mu})$ is sequentially rational. [In PIEFG, every information set is singleton, hence $\widehat{\mu}$ is the degenerate distribution at that singleton.]

Equilibrium with sequential rationality

Definition (Perfect Bayesian Equilibrium)

An assessment (σ, μ) is a *perfect Bayesian equilibrium* (PBE) if *for every player* i

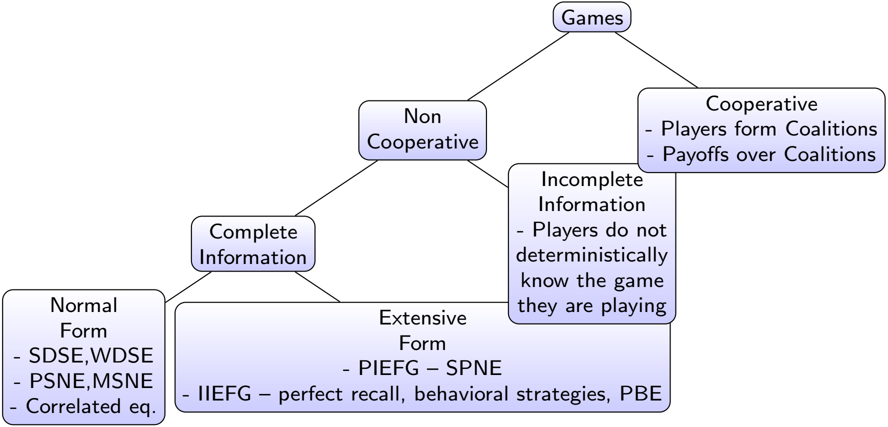
1. μ_i is Bayesian with respect to σ,
2. σ_i is sequentially rational given σ_{-i} and μ_i at every information set of i.

- from the sequential rationality, if μ is Bayesian with respect to σ

Theorem

For every Perfect Bayesian Equilibrium (PBE) (σ, μ), σ *is a Mixed Strategy Nash Equilibrium (MSNE).*

Ecosystem


Games
Non Cooperative
Cooperative
- Players form Coalitions
- Payoffs over Coalitions
Complete Information
Incomplete Information
- Players do not deterministically know the game they are playing
Normal Form
- SDSE,WDSE
- PSNE,MSNE
- Correlated eq.
Extensive Form
- PIEFG – SPNE
- IIEFG – perfect recall, behavioral strategies, PBE