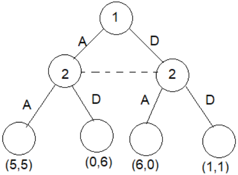
CS711: Introduction to Game Theory and Mechanism Design

Teacher: Swaprava Nath

Imperfect Information Extensive Form Games

Notation



Imperfect-information extensive form game Representation of Neighboring Kingdom Dilemma

1 \ 2	A	D
A	5,5	0,6
D	6,0	1,1

Normal Form Representation

Definition (Imperfect Information Extensive Form Game)

An *imperfect information extensive form game* is a tuple

$$\langle N, A, \mathcal{H}, \mathcal{X}, P, (u_i)_{i \in N}, (I_i)_{i \in N} \rangle$$

Where $\langle N, A, \mathcal{H}, \mathcal{X}, P, (u_i)_{i \in N} \rangle$ is a PIEFG and for every $i \in N$, $I_i := (I_i^1, I_i^2, \ldots, I_i^{k(i)})$ is a partition of $\{h \in \mathcal{H} \setminus Z : P(h) = i\}$ with the property that $\mathcal{X}(h) = \mathcal{X}(h')$ and $P(h) = P(h')$ whenever $\exists j$ s.t. $h, h' \in I_i^j$. The sets in the partition I_i are called **information sets** of player i, and in a specific information set, the actions available to player i are same.

IIEFG (contd.)

- set I_i for every player i, is a collection of information sets $I_i^j, j = 1, \ldots, k(i)$. Information sets are collection of histories where the player at that history is uncertain about which history has been reached.
- the actions at an information set are identical, we can define $\mathcal{X}$ over Information sets I_i^js, rather than defining them over histories h, h'. Therefore

$$\mathcal{X}(h) = \mathcal{X}(h') = \mathcal{X}(I_i^j).$$

- strategies now can be defined over the information sets

Definition (Strategy Set)

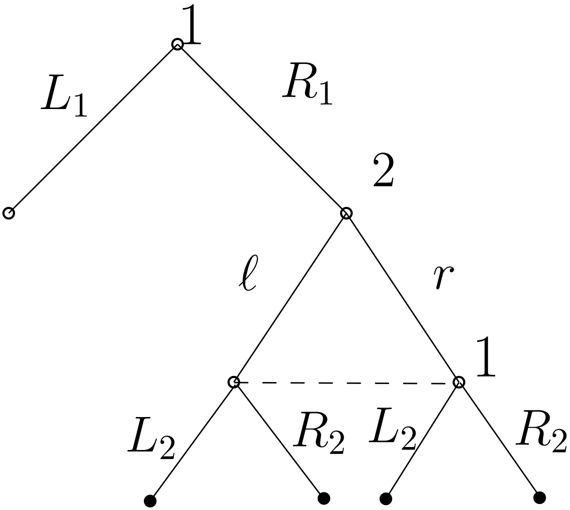
Strategy set of player $i, i \in N$ is defined as the Cartesian product of the actions available to player i at his information sets, i.e.,

$$S_i = \bigtimes_{\tilde{I} \in I_i} \mathcal{X}(\tilde{I}) = \bigtimes_{j=1}^{k(i)} \mathcal{X}(I_i^j).$$

Randomization in IIEFGs

- In NFGs, mixed strategies allows the player to extend her strategy by picking pure strategies randomly
- In EFGs, randomization can happen in different ways
  - player can randomly pick the strategies *defined at the beginning of the game* – mixed strategies
  - player can randomly pick the action at *an information set* – **behavioral strategies**

Example 1



Definition

- behavioral strategy

Definition (Behavioral Strategy)

A *behavioral strategy* of a player in an IIEFG is a function mapping each of her information sets to a probability distribution over the set of possible actions at that information set.

- what is the relation between mixed and behavioral strategies?
- in the example: mixed strategies live in $\mathbb{R}^4$, while behavioral strategies live in two $\mathbb{R}^2$ spaces
- mixed strategies look a "richer" or "larger" concept
- can a player attain higher payoff in one strategy than the other? – contrast the relation between Nash and correrated equilibria

Equivalence of strategies

- the probability of reaching a vertex/history x in the game tree
  - for mixed strategy profile σ: $\rho(x;\sigma)$
  - for behavioral strategy profile b: $\rho(x;b)$
- example from the previous example
- equivalence

Definition

A mixed strategy σ_i and a behavioral strategy b_i of a player i in an IIEFG are *equivalent* if for every mixed/behavioral strategy vector σ_{-i} of the other players and every vertex x in the game tree

$$\rho(x;\sigma_i,\sigma_{-i}) = \rho(x;b_i,\sigma_{-i}).$$

- equivalence of the strategies are defined for a player – other players' strategies can be either mixed or behavioral
- equivalent strategies induce same probability of reaching a vertex

Equivalence of strategies (contd.)

- equivalence holds for leaf nodes in particular
- **claim:** enough to check the equivalence only at the leaf nodes
- **reason:** pick any node, the probability of reaching that node is equal to the sum of the probabilities of reaching the leaf nodes in its subtree
- the argument can be extended further – the utilities at the equivalent strategy vectors yield same payoffs to all players

Theorem (Utility Equivalence)

If a mixed strategy σ_i and a behavioral strategy b_i of a player i are equivalent, then for every mixed/behavioral strategy vector σ_{-i} of the other players and for every player $j \in N$

$$u_j(\sigma_i, \sigma_{-i}) = u_j(b_i, \sigma_{-i}).$$

Equivalence of strategies (contd.)

- the argument of the theorem can be repeated for any equivalent mixed and behavioral strategy profile

Corollary

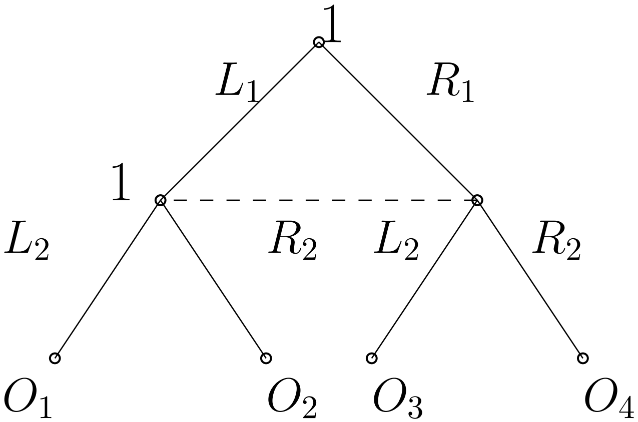
Let $\sigma = (\sigma_i)_{i \in N}$ be a mixed strategy profile. For each player i, let b_i be a behavioral strategy that is equivalent to σ_i. Let $b = (b_i)_{i \in N}$. Then for every $i \in N$

$$u_i(\sigma) = u_i(b).$$

Desirability of behavioral strategies

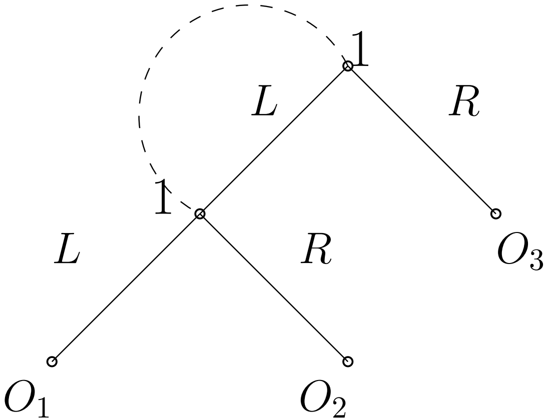
- behavioral strategies are randomization plans at an information set
- this yields the advantage of dealing with smaller number of variables
  - a player having 4 information sets with 2 actions each in every information set
  - needs $2^4 - 1$ variables for mixed strategies
  - only 4 variable for behavioral strategies
- behavioral strategies are *more natural* in large IIEFGs
- players plan at a stage of the game, rather than a master plan
- motivates the question whether we can construct one strategy from the other
- does equivalence always hold?

Example 2



mixed strategy with no equivalent behavioral strategy

Example 3



behavioral strategy with no equivalent mixed strategy

Why this non-equivalence?

- examples 2 and 3 suffer forgetfulness of different kinds
  - example 2: player can remember it made a move but cannot remember what move it made
  - example 3: player cannot remember whether it moved at all
- what happens if the players are not forgetful?
- conditions for equivalence

Mixed strategy equivalent to behavioral strategy

- let x be a non-root node
- terminology: the unique edge emanating from x_1 that is on the path from root to x is the action **at** x_1 **leading to** x
- the trouble of example 3 is that there is a node which has a path from root to itself that crosses the same information set twice
- if the path from root to x passes through vertices x_1 and $\widehat{x}_1$ that are in the same information set of player i, and
- the action leading to x at x_1 is different from the action leading to x at $\widehat{x}_1$, then
- no pure strategy of player i can ever lead to x – see example 3, O_2
- mixed strategy is a probability distribution over the pure strategies – hence every mixed strategy will have zero probability on x and subsequent outcomes
- but behavioral strategies can randomize on every vertex of an information set independently, then x can be reached in a behavioral strategy with positive probability

Equivalence conditions

- a lemma

Lemma

If there exists a path from the root to some vertex x that passes at least twice through the same information set I_i^k of player i, and if the the action leading to x is not the same *action at each of the vertices of the same information set, then player i has a behavioral strategy that has no equivalent mixed strategy.*

- this lemma will help us prove the characterization result for existence of an equivalent mixed strategy of a behavioral strategy
- the theorem

Theorem

Let $\langle N, A, \mathcal{H}, \mathcal{X}, P, (u_i)_{i \in N}, (I_i)_{i \in N} \rangle$ be an IIEFG such that at every vertex of the game tree there are at least two actions. Every behavioral strategy has an equivalent mixed strategy if and only if each information set of player i intersects every path emanating from the root at most once.

Proof: necessity

- **given:** every behavioral strategy of player i has an equivalent mixed strategy
- **to show:** each information set of player i intersects every path emanating from the root at most once
- suppose not, there exists a path from root to x that intersects an information set of i at least twice
- what is the problem?
- each vertex has at least two actions
- there must be a vertex x' such that the the actions leading to x' at two different vertices of the information set of player i are different
- by the previous lemma, then this game has a behavioral strategy of player i that has no equivalent mixed strategy

Proof: sufficiency

- **given:** each information set of player i intersects every path emanating from the root at most once
- **to show:** every behavioral strategy of player i has an equivalent mixed strategy
- let b_i be a behavioral strategy of player i, i.e., it gives a probability distribution over the actions at I_i^k as $b_i(I_i^k)$
- since every information set is cut at most once by any path from root to a node x, player i can pick the sequence of actions leading to x with a probability $=$ the product of all the probabilities given by the behavioral strategies at those information sets
- this sequence of actions with their probabilities can be determined at the beginning of the game, hence it is an **equivalent** mixed strategy
- the construction is not possible if there are two intersections of the path with an information set
- if there were multiple crossings, then the constructed mixed strategy would need the same action to be picked at every crossing, which is restricted than
- what behavioral strategies are capable of doing – it can pick any of the actions independently at every vertex of the information set