

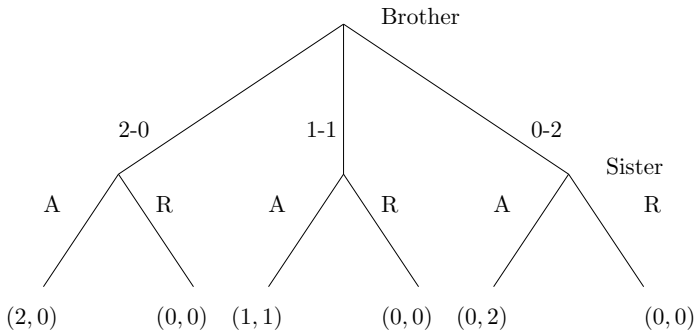
CS711: Introduction to Game Theory and Mechanism Design

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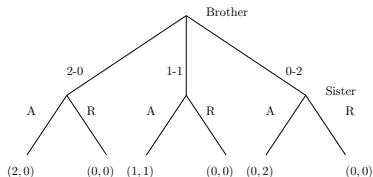
Extensive Form Games

Extensive Form Games

- **Chocolate Division Game:** Suppose a mother gives his elder son two (indivisible) chocolates to share between him and his younger sister. She also warns that if there is any dispute in the sharing, she will take the chocolates back and nobody will get anything. The brother can propose the following sharing options: (2-0): brother gets two, sister gets nothing, or (1-1): both gets one each, or (0-2): both chocolates to the sister. After the brother proposes the sharing, his sister may “Accept” the division or “Reject” it.



Representing the Chocolate Division Game



$$N = \{1 \text{ (brother)}, 2 \text{ (sister)}\}, \quad A = \{2-0, 1-1, 0-2, A, R\}$$

$$\mathcal{H} = \{\emptyset, (2-0), (1-1), (0-2), (2-0, A), (2-0, R), (1-1, A), (1-1, R), (0-2, A), (0-2, R)\}$$

$$Z = \{(2-0, A), (2-0, R), (1-1, A), (1-1, R), (0-2, A), (0-2, R)\}$$

$$\mathcal{X}(\emptyset) = \{(2-0), (1-1), (0-2)\}$$

$$\mathcal{X}(2-0) = \mathcal{X}(1-1) = \mathcal{X}(0-2) = \{A, R\}$$

$$P(\emptyset) = 1, \quad P(2-0) = P(1-1) = P(0-2) = 2$$

$$u_1(2-0, A) = 2, \quad u_1(1-1, A) = 1, \quad u_2(1-1, A) = 1, \quad u_2(0-2, A) = 2$$

$$u_1(0-2, A) = u_1(0-2, R) = u_1(1-1, R) = u_1(2-0, R) = 0$$

$$u_2(0-2, R) = u_2(1-1, R) = u_2(2-0, R) = u_2(2-0, A) = 0$$

$$S_1 = \{2-0, 1-1, 0-2\}$$

$$S_2 = \{A, R\} \times \{A, R\} \times \{A, R\} = \{AAA, AAR, ARA, ARR, RAA, RAR, RRA, RRR\}$$

Representing PIEFG as NFG

- Given S_1 and S_2 , we can represent the game as an NFG, which can be written in the form of matrix.
- For the given example, we can express the utility function as in the following table:

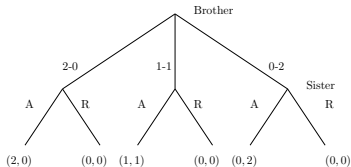
B \ S	AAA	AAR	ARA	ARR	RAA	RAR	RRA	RRR
2-0	(2,0)	(2,0)	(2,0)	(2,0)	(0,0)	(0,0)	(0,0)	(0,0)
1-1	(1,1)	(1,1)	(0,0)	(0,0)	(1,1)	(1,1)	(0,0)	(0,0)
0-2	(0,2)	(0,0)	(0,2)	(0,0)	(0,2)	(0,0)	(0,2)	(0,0)

- Observe that there are many PSNEs in the given game, some of which leads to quite nonintuitive solutions. The PSNEs are marked in **Bold**.

B \ S	AAA	AAR	ARA	ARR	RAA	RAR	RRA	RRR
2-0	(2,0)	(2,0)	(2,0)	(2,0)	(0,0)	(0,0)	(0,0)	(0,0)
1-1	(1,1)	(1,1)	(0,0)	(0,0)	(1,1)	(1,1)	(0,0)	(0,0)
0-2	(0,2)	(0,0)	(0,2)	(0,0)	(0,2)	(0,0)	(0,2)	(0,0)

Existence of PSNE

- We have talked about existence of MSNE, but the equilibrium notion was weaker than PSNE
- Can we find classes of games where PSNE always exist?
- What about the PIEFGs? Any finite PIEFG.



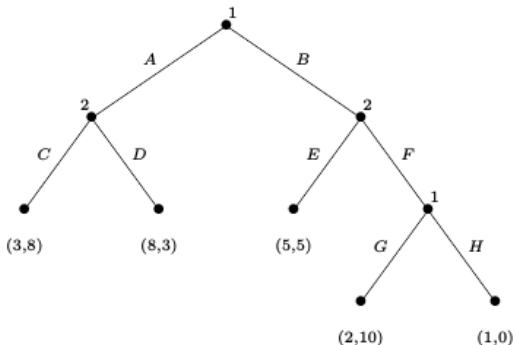
- **Intuition:** at every stage of the game, a player has *perfect information* about the action taken by the former player. Hence there is no reason for *randomizing* over the actions. There is always a pure action that is *weakly superior* than the other actions.
- hence the following result

Theorem

Every finite PIEFG has a PSNE.

But PSNE is a Weak Guarantee

- Consider the PIEFG



- strategies of player 1: AG, AH, BG, BH
- strategies of player 2: CE, CF, DE, DF
- PSNEs: (AG,CF), (AH,CF), (BH,CE)
- non-credible threat** again!
- better notion of rational outcome will be that considers a history and ensures utility maximization for the agent

Subgame and Subgame Perfection

- Subgame is a game rooted at a vertex

Definition (Subgame)

Given a perfect-information extensive-form game G , the subgame of G rooted at node h is the restriction of G to the descendants of h . The set of subgames of G consists of all of subgames of G rooted at some node in G .

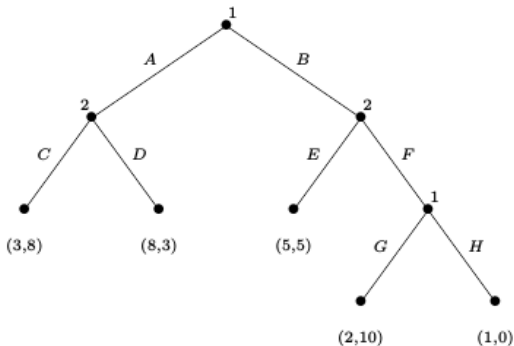
- With subgame, we can define the notion of **subgame perfection**
- Best response at every subgame

Definition (Subgame Perfect Nash Equilibrium)

The *subgame perfect Nash equilibrium* (SPNE) of a game G are all strategy profiles $s \in S := \prod_{i \in N} S_i$ such that for any subgame G' of G the restriction of s to G' is a Nash equilibrium of G' .

Example

- Consider the PIEFG



- strategies of player 1: *AG*, *AH*, *BG*, *BH*
- strategies of player 2: *CE*, *CF*, *DE*, *DF*
- PSNEs: (*AH*, *CF*), (*BH*, *CE*), (*AG*, *CF*)
- are they all SPNE?
- how to find them?

Computing SPNE: Backward Induction

- computation

```
function BACKWARDINDUCTION (node  $h$ ) returns  $u(h)$ 
if  $h \in Z$  then
  return  $u(h)$  //  $h$  is a terminal node
 $best\_util \leftarrow -\infty$ 
forall  $a \in \chi(h)$  do
   $util\_at\_child \leftarrow$  BACKWARDINDUCTION( $\sigma(h, a)$ )
  if  $util\_at\_child_{\rho(h)} > best\_util_{\rho(h)}$  then
    return  $util\_at\_child$ 
return  $best\_util$ 
```

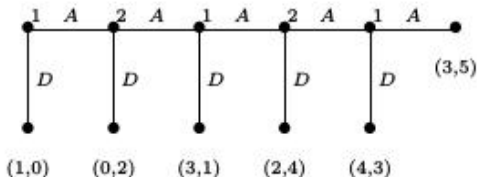
- Inherent in the concept of subgame-perfect equilibrium is the principle of backward induction. One identifies the equilibria in the “bottom-most” subgame trees, and assumes that those equilibria will be played as one backs up and considers increasingly larger trees
- **good news:** not only are we guaranteed to find an SPNE, the algorithm is quite simple
- **bad news:** need to enumerate all possible vertices of the game tree – e.g., for chess, this is around 10^{150} vertices

Implications of Backward Induction and Criticisms

- SPNE: guaranteed to exist for finite PIEFGs (exercise)
- An SPNE is a PSNE of the transformed NFG
- A PSNE exists in the transformed NFG of every finite PIEFG [earlier theorem]
- **Limitations of SPNE:** Centipede game

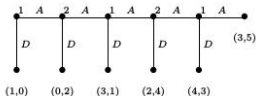
Centipede Game

- In this game two players makes alternate decisions, at each turn choosing between going “down” and ending the game or going “across” and continuing it except at the last node where going “across” also ends the game. The payoffs are constructed in such a way that the player achieves higher payoffs by choosing “down”.



- SPNE of this game?
- what is the problem with that prediction?

Empirical Studies on Centipede



- this game has been experimented with several types of population – general subjects, college students, chess grandmasters
- majority of the experiments conclude that individuals continue till a few rounds of centipede
- it is also rare that players cooperate till the last round
- reason claimed:
 - ▶ **altruism** – people take others on good faith
 - ▶ **limited computational capacity** of the players – does not complete the whole argument of backward induction
 - ▶ **incentives** – if the difference between cooperate and defect are larger, propensity to stop early is also larger
- with a rising Elo, the probability of continuing the game declines
- all Grandmasters in the experiment stopped at their first chance

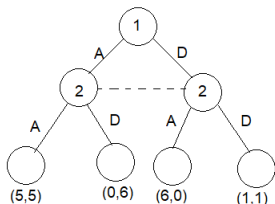
Beyond SPNE

- **theoretical criticism:** the predictions of the longer branches base its hypothesis that 'if the game reaches that point' which is itself contradicted by a later conclusion that 'it cannot reach that point'
- This hints at a situation where players may have a 'soft' prediction of the conclusion – **beliefs**
- **Question:** represent neighboring kingdoms' dilemma with an EFG

A \ B	Agriculture	Defense
Agriculture	5,5	0,6
Defense	6,0	1,1

Imperfect Information Extensive Form Games

- PIEFG is not able to represent the simultaneous move games like neighboring kingdoms' dilemma
- We need the **Imperfect Information Extensive Form Games**

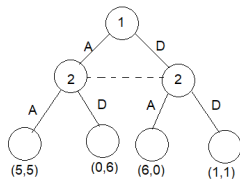


Imperfect-information extensive form game Representation of Neighboring Kingdom Dilemma

1 \ 2	A	D
A	5,5	0,6
D	6,0	1,1

Normal Form Representation

Notation



1 \ 2	A	D
A	5,5	0,6
D	6,0	1,1

Imperfect-information extensive form game Representation of Neighboring Kingdom Dilemma

Normal Form Representation

Definition (Imperfect Information Extensive Form Game)

An *imperfect information extensive form game* is a tuple

$$\langle N, A, \mathcal{H}, \mathcal{X}, P, (u_i)_{i \in N}, (I_i)_{i \in N} \rangle$$

Where $\langle N, A, \mathcal{H}, \mathcal{X}, P, (u_i)_{i \in N} \rangle$ is a PIEFG and for every $i \in N$, $I_i := (I_i^1, I_i^2, \dots, I_i^{k(i)})$ is a partition of $\{h \in \mathcal{H} \setminus Z : P(h) = i\}$ with the property that $\mathcal{X}(h) = \mathcal{X}(h')$ and $P(h) = P(h')$ whenever $\exists j$ s.t. $h, h' \in I_i^j$. The sets in the partition I_i are called **information sets** of player i , and in a specific information set, the actions available to player i are same.

IIEFG (contd.)

- set I_i for every player i , is a collection of information sets $I_i^j, j = 1, \dots, k(i)$. Information sets are collection of histories where the player at that history is uncertain about which history has been reached.
- the actions at an information set are identical, we can define \mathcal{X} over Information sets I_i^j s, rather than defining them over histories h, h' . Therefore

$$\mathcal{X}(h) = \mathcal{X}(h') = \mathcal{X}(I_i^j).$$

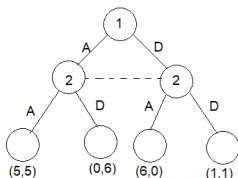
- strategies now can be defined over the information sets

Definition (Strategy Set)

Strategy set of player $i, i \in N$ is defined as the Cartesian product of the actions available to player i at his information sets, i.e.,

$$S_i = \prod_{\tilde{I} \in I_i} \mathcal{X}(\tilde{I}) = \prod_{j=1}^{k(i)} \mathcal{X}(I_i^j).$$

Representations



1 \ 2	A	D
A	5,5	0,6
D	6,0	1,1

Imperfect-information extensive form game Representation of Neighboring Kingdom Dilemma

Normal Form Representation

- NFG can be represented via IIEFG
- but the representation is wasteful – exponentially larger than that of the NFG
- however every NFG can be transformed into an IIEFG, and vice versa – this is an equivalent representation, but clearly one representation is more appropriate for one setting than the other