CS711: Introduction to Game Theory and Mechanism Design

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Correlated Equilibrium, Extensive Form Games

Recap

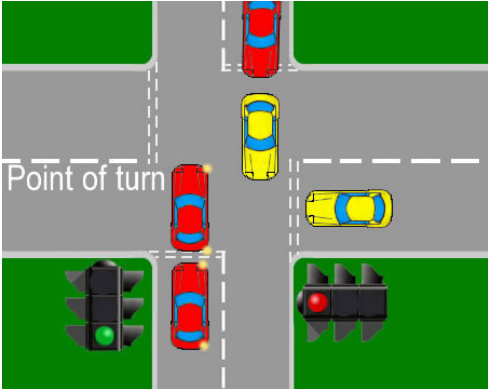
- We have seen games with different types of equilibria and finally arrived at the mixed strategy Nash equilibrium (MSNE) which was the weakest and most general
- The most special property of MSNE is that it always exists for any finite game and can be found by solving a finite number of (potentially non-linear) equations
- However, calculating an MSNE is computationally difficult
- Another equilibrium notion called *correlated equilibrium* (CE) which is weaker than MSNE

Correlated Equilibrium: Motivation

- In a Nash equilibrium, each player chooses his strategy independent of the other player, which may not always lead to the best outcome
- If the players can have a trusted third-party agent, who randomizes over the strategy profiles and suggests the individual strategies to the concerned players, the outcomes can be significantly better
- Such a strategy is called **correlated strategy**
- **Note:** a correlated strategy is *not* a strategy of the players, rather it is a strategy of the third-party agent

Motivation (contd.)

- **Example:** a busy crossing of two roads
  - if the traffic of both roads move at the same time – it is dangerous/chaotic
  - if both stops, it is wasteful
  - mediated movement – trusted third party: traffic lights/police



Definitions

- First we need to define **correlated strategy**

Definition

A *correlated strategy* is a mapping $\pi : S \mapsto [0,1]$ such that $\sum_{s \in S} \pi(s) = 1$ where $S = S_1 \times S_2 \times \cdots \times S_n$ and S_i represents the strategy set of player i.

- Hence, a correlated strategy π is a joint probability distribution over the strategy profiles
- A correlated strategy is a **correlated equilibrium** if it becomes self-enforcing, i.e., no player 'gains' by deviating from the suggested strategy
- **Note:** Here the suggested strategy π is a common knowledge

Definition

A *correlated equilibrium* (CE) is a correlated strategy π such that $\forall s_i \in S_i$ and $\forall i \in N$,

$$\sum_{s_{-i} \in S_{-i}} \pi(s_i, s_{-i}) u_i(s_i, s_{-i}) \geqslant \sum_{s_{-i} \in S_{-i}} \pi(s_i, s_{-i}) u_i(s_i', s_{-i}) \qquad \forall s_i' \in S_i. \tag{1}$$

Comments on CE

- Player i does not gain any advantage in the expected utility if he deviates from the suggested correlated strategy π, assuming all other players follow the suggested strategy.
- Another way to interpret a correlated equilibrium is that π is a single randomization device (a dice e.g.) which gives a random outcome which is a strategy profile, and a specific player only observes the strategy corresponding to her.
- Given that observation, she computes her expected utility, and if that does not improve if she picks another strategy (and it happens for every player) then that randomization device is a correlated equilibrium.
- Few examples to follow

Example 1: Game Selection Problem

- Cricket or Football game

1 \2	C	F
C	2,1	0,0
F	0,0	1,2

- In MSNE, we saw that the expected utility of each player was $\frac{2}{3}$
- Is $\pi(C, C) = \frac{1}{2} = \pi(F, F)$ a CE?

Example 2: Crossroad Collision Problem

- Chicken game

1 \2	Stop	Go
Stop	0,0	1,2
Go	2,1	-10,-10

- Consider a correlated strategy π such that $\pi(S,G) = \pi(S,S) = \pi(G,S) = \frac{1}{3}$
- Is this a CE?
- Other CEs?

Another Interpretation

- Another way to interpret a CE is that it is a distribution over the strategy profiles such that if a strategy which has a non-zero probability of occurrence for Player i is suggested to i, the player can compute the posterior distribution of the strategies suggested to other players
- Player i's expected payoff according to that distribution will be maximized by following the suggestion if other players follow their respective suggestions as well. More formally, let $\bar{s}_i$ be the strategy suggested to Player i, then it is a CE if $\forall i \in N$:

$$\sum_{s_{-i} \in S_{-i}} \pi(s_{-i}|\bar{s}_i) u_i(\bar{s}_i, s_{-i}) \geqslant \sum_{s_{-i} \in S_{-i}} \pi(s_{-i}|\bar{s}_i) u_i(s'_i, s_{-i}), \ \forall s'_i \in S_i$$

$$\Rightarrow \sum_{s_{-i} \in S_{-i}} \pi(\bar{s}_i, s_{-i}) u_i(\bar{s}_i, s_{-i}) \geqslant \sum_{s_{-i} \in S_{-i}} \pi(\bar{s}_i, s_{-i}) u_i(s'_i, s_{-i}), \ \forall s'_i \in S_i.$$

Computing the Correlated Equilibrium

- To find a CE, we need to solve a set of linear equations with the variables as $\pi(s), s \in S$
- $\pi(s)$ is a CE if $\forall s_i \in S_i$, and $\forall i \in N$

$$\sum_{s_{-i} \in S_{-i}} \pi(s) u_i(s_i, s_{-i}) \geq \sum_{s_{-i} \in S_{-i}} \pi(s) u_i(s_i', s_{-i}) \forall s_i' \in S_i. \qquad (2)$$

- The total number of inequalities here are $O(nm^2)$, assuming $|S_i| = m, \forall i \in N$
- We also need to ensure that $\pi(s)$ is a valid probability distribution. Therefore

$$\pi(s) \geq 0, \ \forall s \in S \qquad m^n \text{ inequalities} \qquad (3)$$

$$\sum_{s \in S} \pi(s) = 1 \qquad (4)$$

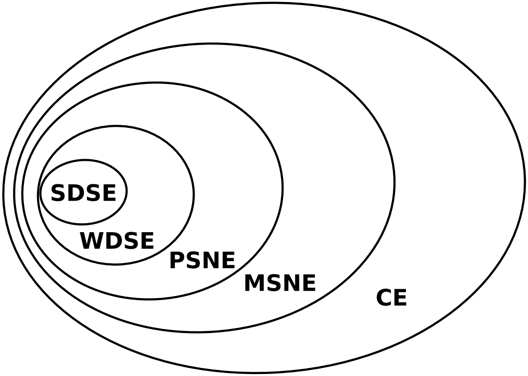
- The inequalities together represent a feasibility LP which is easier to compute than an MSNE
- For computing MSNE, the number of support profiles are $O(2^{mn})$, which is exponentially larger than the number of inequalities to find a CE $(O(m^n))$. Therefore computing a CE is a much simpler problem than a MSNE.

Finishing the Equilibrium Space

Theorem

For every mixed strategy Nash equilibrium σ^*, there exists a correlated equilibrium π^* of the same game

Proof hint: $\pi^*(s_1, \ldots, s_n) = \prod_{i \in N} \sigma_i^*(s_i)$.



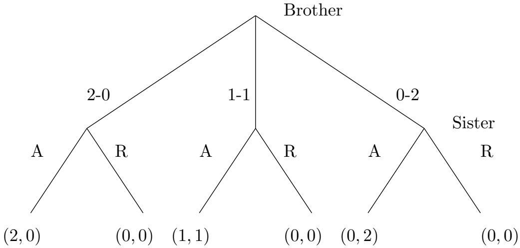
Summary of Normal Form Games

1. basic definitions of normal form games
2. **rationality**, **intelligence**, **common knowledge**
3. **strategy** and **action**
4. **dominance** – strict and weak
5. **SDSE, WDSE** – examples of (non)existence
6. unilateral deviation from a profile – **PSNE**
7. nonexistence of PSNE – **MSNE**
8. existence guarantee of MSNE – Nash's theorem
9. characterization result of MSNE
10. computational hardness and more practical considerations
11. trusted mediator/third party – correlated strategies
12. **correlated equilibrium** – computational efficiency

Extensive Form Games

Extensive Form Games

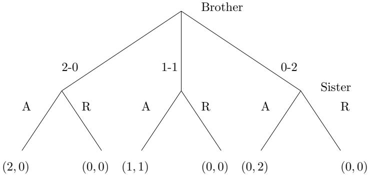
- Representation format more appropriate for multi-stage games
- Players interact in a sequence and the sequence of actions is called **history**
- **Perfect Information Extensive Form Games** (PIEFG) – every player observes the exact moves of the other player
- **Chocolate Division Game:** Suppose a mother gives his elder son two (indivisible) chocolates to share between him and his younger sister. She also warns that if there is any dispute in the sharing, she will take the chocolates back and nobody will get anything. The brother can propose the following sharing options: (2-0): brother gets two, sister gets nothing, or (1-1): both gets one each, or (0-2): both chocolates to the sister. After the brother proposes the sharing, his sister may “Accept” the division or “Reject” it.



Notation

- We formally denote a PIEFG by the tuple $\langle N, A, \mathcal{H}, \mathcal{X}, P, (u_i)_{i \in N} \rangle$, where
  - N : set of players
  - A : set of all possible actions (of all players)
  - $\mathcal{H}$: set of all **sequences of actions** (histories) satisfying
    - empty sequence $\varnothing \in \mathcal{H}$
    - if $h \in \mathcal{H}$, any initial continuous sub-sequence h' of h belongs to $\mathcal{H}$
    - a history $h = (a^{(0)}, a^{(1)}, \dots, a^{(T-1)})$ is **terminal** if $\nexists\ a^{(T)} \in A$ s.t. $(a^{(0)}, a^{(1)}, \dots, a^{(T-1)}, a^{(T)}) \in \mathcal{H}$
  - $Z \subseteq \mathcal{H}$: set of all *terminal* histories
  - $\mathcal{X} : \mathcal{H} \setminus Z \mapsto 2^A$: action set selection function
  - $P : \mathcal{H} \setminus Z \mapsto N$: player function
  - $u_i : Z \mapsto \mathbb{R}$: utility function of player i
- The **strategy** of a player in an EFG is a sequence of actions at every history where the player plays. Formally $S_i = \times_{\{h \in \mathcal{H} : P(h) = i\}} \mathcal{X}(h)$
- It is a **complete contingency plan** of the player. It enumerates potential actions a player can take at every node where he can play, even though some sequence of actions may never be executed together.

Representing the Chocolate Division Game



$$N = \{1 \text{ (brother)}, 2 \text{ (sister)}\}, \; A = \{2-0, 1-1, 0-2, A, R\}$$
$$\mathcal{H} = \{\emptyset, (2-0), (1-1), (0-2), (2-0, A), (2-0, R), (1-1, A), (1-1, R), (0-2, A), (0-2, R)\}$$
$$Z = \{(2-0, A), (2-0, R), (1-1, A), (1-1, R), (0-2, A), (0-2, R)\}$$
$$\mathcal{X}(\emptyset) = \{(2-0), (1-1), (0-2)\}$$
$$\mathcal{X}(2-0) = \mathcal{X}(1-1) = \mathcal{X}(0-2) = \{A, R\}$$
$$P(\emptyset) = 1, \; P(2-0) = P(1-1) = P(0-2) = 2$$
$$u_1(2-0, A) = 2, \; u_1(1-1, A) = 1, \; u_2(1-1, A) = 1, \; u_2(0-2, A) = 2$$
$$u_1(0-2, A) = u_1(0-2, R) = u_1(1-1, R) = u_1(2-0, R) = 0$$
$$u_2(0-2, R) = u_2(1-1, R) = u_2(2-0, R) = u_2(2-0, A) = 0$$
$$S_1 = \{2-0, 1-1, 0-2\}$$
$$S_2 = \{A, R\} \times \{A, R\} \times \{A, R\} = \{AAA, AAR, ARA, ARR, RAA, RAR, RRA, RRR\}$$

Representing PIEFG as NFG

- Given S_1 and S_2, we can represent the game as an NFG, which can be written in the form of matrix.
- For the given example, we can express the utility function as in the following table:

B \ S	AAA	AAR	ARA	ARR	RAA	RAR	RRA	RRR
2-0	(2,0)	(2,0)	(2,0)	(2,0)	(0,0)	(0,0)	(0,0)	(0,0)
1-1	(1,1)	(1,1)	(0,0)	(0,0)	(1,1)	(1,1)	(0,0)	(0,0)
0-2	(0,2)	(0,0)	(0,2)	(0,0)	(0,2)	(0,0)	(0,2)	(0,0)

- Observe that there are many PSNEs in the given game, some of which leads to quite nonintuitive solutions. The PSNEs are marked in **Bold**.

B \ S	AAA	AAR	ARA	ARR	RAA	RAR	RRA	RRR
2-0	**(2,0)**	**(2,0)**	**(2,0)**	**(2,0)**	(0,0)	(0,0)	**(0,0)**	**(0,0)**
1-1	(1,1)	(1,1)	(0,0)	(0,0)	**(1,1)**	**(1,1)**	(0,0)	(0,0)
0-2	(0,2)	(0,0)	(0,2)	(0,0)	(0,2)	(0,0)	**(0,2)**	(0,0)

- PSNE like {2-0,RRA} is not a reasonable guarantee and {2-0,RRR} is not a **credible threat** – PSNE is not good enough for this game
- The representation is very wasteful and the EFG representation is succinct for such cases