CS711: Introduction to Game Theory and Mechanism Design

Teacher: Swaprava Nath

Mixed Strategies, Nash Theorem

Proof of the Characterization Theorem

Theorem (Characterization of a MSNE)

A mixed strategy profile $(\sigma_i^*, \sigma_{-i}^*)$ *is a MSNE iff* $\forall i \in N$

1. $u_i(s_i, \sigma_{-i}^*)$ *is the same for all* $s_i \in \delta(\sigma_i^*)$, *and*
2. $u_i(s_i, \sigma_{-i}^*) \geq u_i(s_i', \sigma_{-i}^*), \ \forall \ s_i \in \delta(\sigma_i^*), s_i' \notin \delta(\sigma_i^*)$.

- Observations:

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- a maximizer s_i must lie in $\delta(\sigma_i^*)$ – if none of the maximizers live in $\delta(\sigma_i^*)$, then one can construct a mixed strategy by placing all mass on that $s_i' \notin \delta(\sigma_i^*)$ which will be strictly better than the utility at the MSNE – a contradiction

Proof (contd.)

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- by definition of expected utility for the given strategy profile we have

$$\begin{aligned} u_i(\sigma_i^*, \sigma_{-i}^*) &= \sum_{s_i \in S_i} \sigma_i^*(s_i) \cdot u_i(s_i, \sigma_{-i}^*) \\ &= \sum_{s_i \in \delta(\sigma_i^*)} \sigma_i^*(s_i) \cdot u_i(s_i, \sigma_{-i}^*) \end{aligned} \tag{2}$$

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- Equating the 1 and 2: expectation and the maximum value of a set are equal – happen only when either the set is singleton or all the elements take the same value – condition 1 proved

Proof (contd.)

- to prove condition 2: suppose for contradiction

$$\exists s_i \in \delta(\sigma_i^*), s_i' \notin \delta(\sigma_i^*) \text{ s.t. } u_i(s_i, \sigma_{-1}^*) < u_i(s_i', \sigma_{-i}^*)$$

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- using condition 2, we conclude $m_i(\sigma_{-i}^*) = \max_{s_i \in S_i} u_i(s_i, \sigma_{-i}^*)$

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$$u_i(\sigma_i^*, \sigma_{-i}^*) = \sum_{s_i \in \delta(\sigma_i^*)} \sigma_i^*(s_i) \cdot u_i(s_i, \sigma_{-i}^*) \tag{3}$$

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[1] Daskalakis, Constantinos, Paul W. Goldberg, and Christos H. Papadimitriou. "The complexity of computing a Nash equilibrium." SIAM Journal on Computing 39.1 (2009): 195-259.

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- linear if $n = 2$, otherwise non-linear
- for general games, there is no known poly-time algorithm

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Algorithm to find MSNE (contd.)

- feasibility program

$$w_i = \sum_{s_{-i} \in S_{-i}} \left(\prod_{j \neq i} \sigma_j(s_j) \right) u_i(s_i, s_{-i}), \forall s_i \in X_i, \forall i \in N$$

$$w_i \geqslant \sum_{s_{-i} \in S_{-i}} \left(\prod_{j \neq i} \sigma_j(s_j) \right) u_i(s_i, s_{-i}), \forall s_i \in S_i \setminus X_i, \forall i \in N$$

$$\sigma_j(s_j) \geqslant 0, s_j \in S_j, j \in N, \text{ and } \sum_{s_j \in S_j} \sigma_j(s_j) = 1, \forall j \in N$$

variables $w_i, i \in N$, $\sigma_j(s_j), s_j \in S_j, j \in N$

- linear if $n = 2$, otherwise non-linear
- for general games, there is no known poly-time algorithm
- problem of finding a MSNE is PPAD complete – Daskalakis et al. (2009)[1]

[1] Daskalakis, Constantinos, Paul W. Goldberg, and Christos H. Papadimitriou. "The complexity of computing a Nash equilibrium." SIAM Journal on Computing 39.1 (2009): 195-259.

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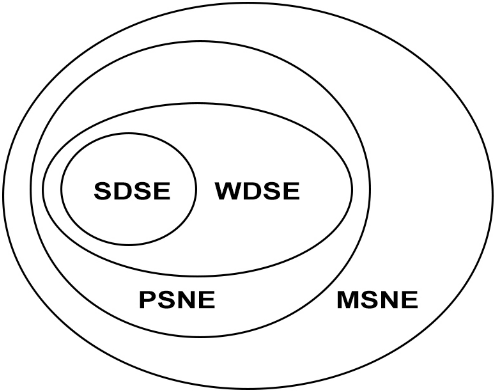

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- Hence s_i can be removed without loss of equilibria

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Theorem (Brouwer's Fixed Point Theorem)

If $S \subseteq \mathbb{R}^n$ is convex and compact and $T : S \mapsto S$ is continuous, then T has a fixed point, i.e., $\exists$ a point $x^* \in S$ s.t. $T(x^*) = x^*$.