CS711: Introduction to Game Theory and Mechanism Design

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Mixed Strategies, Nash Theorem

Proof of the Characterization Theorem

Theorem (Characterization of a MSNE)

A mixed strategy profile $(\sigma_i^*, \sigma_{-i}^*)$ *is a MSNE iff* $\forall i \in N$

1. $u_i(s_i, \sigma_{-i}^*)$ *is the same for all* $s_i \in \delta(\sigma_i^*)$*, and*
2. $u_i(s_i, \sigma_{-i}^*) \geq u_i(s_i', \sigma_{-i}^*), \ \forall \ s_i \in \delta(\sigma_i^*), s_i' \notin \delta(\sigma_i^*)$.

- Observations:
- first:

$$\max_{\sigma_i \in \Delta(S_i)} u_i(\sigma_i, \sigma_{-i}^*) = \max_{s_i \in S_i} u_i(s_i, \sigma_{-i}^*)$$

- maximizing w.r.t. a distribution $\equiv$ whole probability mass at the maximum value (or splits arbitrarily over the maximum values)
- second

$$\max_{\sigma_i \in \Delta(S_i)} u_i(\sigma_i, \sigma_{-i}^*) = \max_{s_i \in S_i} u_i(s_i, \sigma_{-i}^*) = \max_{s_i \in \delta(\sigma_i^*)} u_i(s_i, \sigma_{-i}^*)$$

- a maximizer s_i must lie in $\delta(\sigma_i^*)$ – if none of the maximizers live in $\delta(\sigma_i^*)$, then one can construct a mixed strategy by placing all mass on that $s_i' \notin \delta(\sigma_i^*)$ which will be strictly better than the utility at the MSNE – a contradiction

Proof (contd.)

- **Proof:** $(\Rightarrow)$ given $(\sigma_i^*, \sigma_{-i}^*)$ is an MSNE, the two conditions hold
- given $(\sigma_i^*, \sigma_{-i}^*)$ is an MSNE

$$\begin{aligned} u_i(\sigma_i^*, \sigma_{-i}^*) &= \max_{\sigma_i \in \Delta(S_i)} u_i(\sigma_i, \sigma_{-i}^*) \\ &= \max_{s_i \in S_i} u_i(s_i, \sigma_{-i}^*) \\ &= \max_{s_i \in \delta(\sigma_i^*)} u_i(s_i, \sigma_{-i}^*) \end{aligned} \tag{1}$$

- by definition of expected utility for the given strategy profile we have

$$\begin{aligned} u_i(\sigma_i^*, \sigma_{-i}^*) &= \sum_{s_i \in S_i} \sigma_i^*(s_i) \cdot u_i(s_i, \sigma_{-i}^*) \\ &= \sum_{s_i \in \delta(\sigma_i^*)} \sigma_i^*(s_i) \cdot u_i(s_i, \sigma_{-i}^*) \end{aligned} \tag{2}$$

- Equating the 1 and 2: expectation and the maximum value of a set are equal – happen only when either the set is singleton or all the elements take the same value – condition 1 proved

Proof (contd.)

- to prove condition 2: suppose for contradiction

$$\exists s_i \in \delta(\sigma_i^*), s_i' \notin \delta(\sigma_i^*) \text{ s.t. } u_i(s_i, \sigma_{-1}^*) < u_i(s_i', \sigma_{-i}^*)$$

- transfer all the mass of $\sigma_i^*(s_i)$ to s_i' – this new mixed strategy will yield a strictly better utility – contradiction to MSNE
- $(\Leftarrow)$ given the two conditions of the characterization theorem hold
- define $u_i(s_i, \sigma_{-i}^*) =: m_i(\sigma_{-i}^*)$, for all $s_i \in \delta(\sigma_i^*)$ – possible to define due to condition 1
- using condition 2, we conclude $m_i(\sigma_{-i}^*) = \max_{s_i \in S_i} u_i(s_i, \sigma_{-i}^*)$

Proof (contd.)

$$
\begin{aligned}
u_i(\sigma_i^*, \sigma_{-i}^*) &= \sum_{s_i \in \delta(\sigma_i^*)} \sigma_i^*(s_i) \cdot u_i(s_i, \sigma_{-i}^*) \\
&= m_i(\sigma_{-i}^*) \\
&= \max_{s_i \in S_i} u_i(s_i, \sigma_{-i}^*) \\
&= \max_{\sigma_i \in \Delta(S_i)} u_i(\sigma_i, \sigma_{-i}^*) \\
&\geq u_i(\sigma_i, \sigma_{-1}^*) \quad \forall \sigma_i \in \Delta(S_i)
\end{aligned}
\tag{3}
$$

- first equality holds by definition of $\delta(\sigma_i^*)$
- next two equalities hold due to conditions (1) and (2) as explained before
- last equality is by the observation
- $\implies$ $(\sigma_i^*, \sigma_{-i}^*)$ is an MSNE

Summary: this theorem gives an algorithm to find an MSNE
Question: is this algorithm guaranteed to yield an outcome? Yes!

Algorithm to find MSNE

- NFG $G = \langle N, (S_i)_{i \in N}, (u_i)_{i \in N} \rangle$
- all possible supports of $S_1 \times S_2 \times \cdots \times S_n$
- $K = (2^{|S_1|} - 1) \times (2^{|S_2|} - 1) \times \cdots \times (2^{|S_n|} - 1)$
- given a support profile $X_1 \times X_2 \times \cdots \times X_n$, where $X_i \subseteq S_i$
- solve the following feasibility program

$$w_i = \sum_{s_{-i} \in S_{-i}} \left(\prod_{j \neq i} \sigma_j(s_j) \right) u_i(s_i, s_{-i}), \forall s_i \in X_i, \forall i \in N$$

$$w_i \geqslant \sum_{s_{-i} \in S_{-i}} \left(\prod_{j \neq i} \sigma_j(s_j) \right) u_i(s_i, s_{-i}), \forall s_i \in S_i \setminus X_i, \forall i \in N$$

$$\sigma_j(s_j) \geqslant 0, s_j \in S_j, j \in N, \text{ and } \sum_{s_j \in S_j} \sigma_j(s_j) = 1, \forall j \in N$$

variables $w_i, i \in N$, $\sigma_j(s_j), s_j \in S_j, j \in N$

Algorithm to find MSNE (contd.)

- feasibility program

$$w_i = \sum_{s_{-i} \in S_{-i}} \left(\prod_{j \neq i} \sigma_j(s_j) \right) u_i(s_i, s_{-i}), \forall s_i \in X_i, \forall i \in N$$

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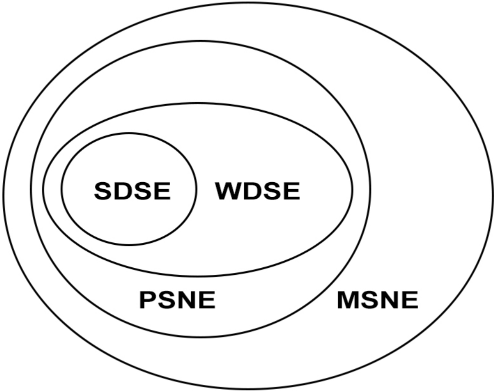
$$\sigma_j(s_j) \geqslant 0, s_j \in S_j, j \in N, \text{ and } \sum_{s_j \in S_j} \sigma_j(s_j) = 1, \forall j \in N$$

  variables $w_i, i \in N$, $\sigma_j(s_j), s_j \in S_j, j \in N$
- linear if $n = 2$, otherwise non-linear
- for general games, there is no known poly-time algorithm
- problem of finding a MSNE is PPAD complete – Daskalakis et al. (2009)[1]

[1] Daskalakis, Constantinos, Paul W. Goldberg, and Christos H. Papadimitriou. "The complexity of computing a Nash equilibrium." SIAM Journal on Computing 39.1 (2009): 195-259.

Relation between the Equilibria Concepts

- Equilibria concepts discussed: SDSE, WDSE, PSNE, MSNE
- What is the relationship – expressed as implications?
- Let (s_i^*, s_{-i}^*) is an SDSE
- Hence the utility is strictly better at s_i^* for i for every s_{-i}
- Implies WDSE
- Extend similar arguments for the rest of the equilibria



MSNE and Dominance

- The previous algorithm can be applied to a smaller set of strategies by removing the dominated strategies
- Is there any dominated strategy for any player

1**2**	L	R
T	4,1	2,5
M	1,3	6,2
B	2,2	3,3

- **domination can be by another pure strategy or a mixed strategy**
- but for weakly dominated strategies it may remove certain equilibria too
- for the strictly dominated strategies, the following holds

Theorem

Consider an NFG $G = \langle N, (S_i)_{i \in N}, (u_i)_{i \in N} \rangle$. If a pure strategy s_i is strictly dominated by a mixed strategy $\sigma_i \in \Delta(S_i)$, then in every MSNE of the game, the pure strategy s_i is chosen with probability 0.

- Hence s_i can be removed without loss of equilibria

Existence of MSNE

- **Finite game:** A game in which the number of players and the strategies are finite.

Theorem (Nash (1951))

Every finite game has a (mixed) Nash equilibrium.

- proof needs a few definitions and a known result
  - A set $S \subseteq \mathbb{R}^n$ is **convex** if $\forall x, y \in S$ and $\forall \lambda \in [0, 1]$, $\lambda x + (1 - \lambda) y \in S$.
  - A set $S \subseteq \mathbb{R}^n$ is **closed** if it contains all its limit points (points whose every neighborhood contains a point in S – e.g., for the point 1 in the interval $[0, 1)$, consider a ball of radius $\epsilon > 0$, arbitrary, clearly, each such ball will contain a point in $[0, 1)$).
  - A set $S \subseteq \mathbb{R}^n$ is **bounded** if $\exists \ x_0 \in \mathbb{R}^n$ and $R \in (0, \infty)$ such that $\forall x \in S, \| x - x_0 \|_2 < R$.
  - A set $S \subseteq \mathbb{R}^n$ is **compact** if it is *closed* and *bounded*.
- a result from real analysis without proof

Theorem (Brouwer's Fixed Point Theorem)

If $S \subseteq \mathbb{R}^n$ is convex and compact and $T : S \mapsto S$ is continuous, then T has a fixed point, i.e., $\exists$ a point $x^* \in S$ s.t. $T(x^*) = x^*$.