CS711: Introduction to Game Theory and Mechanism Design

Teacher: Swaprava Nath

Mixed Strategies

Recap

- iterated elimination of dominated strategies

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- preservation of equilibrium

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- situation where stability and security coincide – two player zero sum games

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- preservation of equilibrium
- situation where stability and security coincide – two player zero sum games
- discussions limited to pure strategies – but an equilibrium may not exist

Penalty shootout game

S**G**	L	R
L	-1,1	1,-1
R	1,-1	-1,1

Mixed Strategies

- probability distribution over the set of strategies

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overload of the notation u_i

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- formal proof left as exercise

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Definition (Mixed Strategy Nash Equilibrium)

MSNE is a mixed strategy profile $(\sigma_i^*, \sigma_{-i}^*)$ s.t.

$$u_i(\sigma_i^*, \sigma_{-i}^*) \geq u_i(\sigma_i', \sigma_{-i}^*) \; \forall \sigma_i' \in \Delta(S_i), \; \forall i \in N.$$

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Theorem

Consider an NFG $G = \langle N, (S_i)_{i \in N}, (u_i)_{i \in N} \rangle$. A mixed strategy profile $(\sigma_i^*, \sigma_{-i}^*)$ is an MSNE if and only if

$$u_i(\sigma_i^*, \sigma_{-i}^*) \geqslant u_i(s_i, \sigma_{-i}^*) \ \forall s_i \in S_i, \ \forall i \in N.$$

Proof

- Proof: $(\Rightarrow)$ s_i is a special case of the mixed strategy, where the mixed strategy of player i is degenerate at s_i, hence the inequality holds by definition of MSNE

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$$\begin{aligned} u_i(\sigma_i, \sigma^*_{-i}) &= \sum_{s_i \in S_i} \sigma_i(s_i) u_i(s_i, \sigma^*_{-i}) \\ &\leqslant \sum_{s_i \in S_i} \sigma_i(s_i) u_i(\sigma^*_i, \sigma^*_{-i}) \\ &= u_i(\sigma^*_i, \sigma^*_{-i}) \sum_{s_i \in S_i} \sigma_i(s_i) \\ &= u_i(\sigma^*_i, \sigma^*_{-i}) \end{aligned}$$

Example of MSNE

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- Repeat the calculations

Finding an MSNE

- **support of a mixed strategy**

Finding an MSNE

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  - for player 2, $s_2 = T$ violates condition 2
- **support = ($\{\mathbf{H}\}$,$\{\mathbf{H},\mathbf{T}\}$)**
  - the expected utility for player 2 has to be equal for H and T – cannot happen – violates condition 1

Application of Characterization (contd.)

- **support** $= (\{\mathbf{H,T}\},\{\mathbf{H,T}\})$

Application of Characterization (contd.)

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  - condition 1 for player 2: $u_2((p,1-p),H) = u_2((p,1-p),T)$, gives the value of p
  - hence the MSNE for this game $((\frac{1}{2},\frac{1}{2}),(\frac{1}{2},\frac{1}{2}))$

Two more exercises (in class)

- Football or Cricket game

$\mathbf{1} \backslash \mathbf{2}$	F	C
F	2,1	0,0
C	0,0	1,2

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- An arbitrary added strategy for player 2

1 \ 2	F	C	D
F	2,1	0,0	1,1
C	0,0	1,2	2,0