CS711: Introduction to Game Theory and Mechanism Design

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Mixed Strategies

Recap

- iterated elimination of dominated strategies
- preservation of equilibrium
- situation where stability and security coincide – two player zero sum games
- discussions limited to pure strategies – but an equilibrium may not exist

Penalty shootout game

S**G**	L	R
L	-1,1	1,-1
R	1,-1	-1,1

Mixed Strategies

- probability distribution over the set of strategies

	S\G	$\frac{4}{5}$ L	$\frac{1}{5}$ R
$\frac{2}{3}$	L	-1,1	1,-1
$\frac{1}{3}$	R	1,-1	-1,1

- **mixed strategy** of a player
  - finite set A, $\Delta(A)$ is defined as the set of all probability distributions over A – i.e., $\Delta(A) = \{p \in [0,1]^{|A|} : \sum_{a \in A} p(a) = 1\}$
  - σ_i is a mixed strategy of player i
  - $\sigma_i \in \Delta(S_i)$ which implies
  - $\sigma_i : S_i \mapsto [0,1]$ s.t. $\sum_{s_i \in S_i} \sigma_i(s_i) = 1$
- non-cooperative game – players pick their strategies independently
- the joint probability that player 1 picks s_1 and 2 picks $s_2 = \sigma_1(s_1) \times \sigma_2(s_2)$
- Utility of player i at a mixed strategy profile (σ_i, σ_{-i}) is

$$u_i(\sigma_i, \sigma_{-i}) = \sum_{(s_1,\ldots,s_n) \in S_1 \times \ldots \times S_n} (\sigma_1(s_1) \times \ldots \times \sigma_n(s_n))\, u_i(s_1, \ldots, s_n)$$

overload of the notation u_i

Examples

	S\G	$\frac{4}{5}$ L	$\frac{1}{5}$ R
$\frac{2}{3}$	L	-1,1	1,-1
$\frac{1}{3}$	R	1,-1	-1,1

- $\sigma_1 = (\frac{2}{3}, \frac{1}{3})$ and $\sigma_2 = (\frac{4}{5}, \frac{1}{5})$
- $u_1(\sigma_1, \sigma_2) =$
- $\frac{2}{3} \cdot \frac{4}{5} \cdot (-1) + \frac{2}{3} \cdot \frac{1}{5} \cdot (1) + \frac{1}{3} \cdot \frac{4}{5} \cdot (1) + \frac{1}{3} \cdot \frac{1}{5} \cdot (-1)$
- $u_2(\sigma_1, \sigma_2) = \frac{2}{3} \cdot \frac{4}{5} \cdot (1) + \frac{2}{3} \cdot \frac{1}{5} \cdot (-1) + \frac{1}{3} \cdot \frac{4}{5} \cdot (-1) + \frac{1}{3} \cdot \frac{1}{5} \cdot (1)$
- utility at a mixed strategy profile (σ_1, σ_2) is an expected utility taken w.r.t. the probability distribution $\sigma_1 \cdot \sigma_2$
- all the properties of expectation holds – e.g., linearity
- utility at the strategy profile $(\lambda\sigma_i + (1-\lambda)\sigma_i', \sigma_{-i})$ is given by

$$u_i(\lambda\sigma_i + (1-\lambda)\sigma_i', \sigma_{-i}) = \lambda u_i(\sigma_i, \sigma_{-i}) + (1-\lambda) u_i(\sigma_i', \sigma_{-i}), \forall \sigma_{-i} \in \bigtimes_{j \neq i} \Delta(S_j)$$

- formal proof left as exercise

Mixed Strategy Nash Equilibrium

Definition (Mixed Strategy Nash Equilibrium)

MSNE is a mixed strategy profile $(\sigma_i^*, \sigma_{-i}^*)$ s.t.

$$u_i(\sigma_i^*, \sigma_{-i}^*) \geq u_i(\sigma_i', \sigma_{-i}^*) \; \forall \sigma_i' \in \Delta(S_i), \; \forall i \in N.$$

- relation of PSNE and MSNE?
- PSNE $\implies$ MSNE
- equivalence of the definition with pure strategies on the RHS

Theorem

Consider an NFG $G = \langle N, (S_i)_{i \in N}, (u_i)_{i \in N} \rangle$. A mixed strategy profile $(\sigma_i^*, \sigma_{-i}^*)$ is an MSNE if and only if

$$u_i(\sigma_i^*, \sigma_{-i}^*) \geqslant u_i(s_i, \sigma_{-i}^*) \; \forall s_i \in S_i, \; \forall i \in N.$$

Proof

- Proof: $(\Rightarrow)$ s_i is a special case of the mixed strategy, where the mixed strategy of player i is degenerate at s_i, hence the inequality holds by definition of MSNE
- $(\Leftarrow)$ pick any arbitrary mixed strategy σ_i for i
- We can write

$$\begin{aligned} u_i(\sigma_i, \sigma^*_{-i}) &= \sum_{s_i \in S_i} \sigma_i(s_i) u_i(s_i, \sigma^*_{-i}) \\ &\leqslant \sum_{s_i \in S_i} \sigma_i(s_i) u_i(\sigma^*_i, \sigma^*_{-i}) \\ &= u_i(\sigma^*_i, \sigma^*_{-i}) \sum_{s_i \in S_i} \sigma_i(s_i) \\ &= u_i(\sigma^*_i, \sigma^*_{-i}) \end{aligned}$$

Example of MSNE

	S\G	$\frac{4}{5}$ L	$\frac{1}{5}$ R
$\frac{2}{3}$	L	-1,1	1,-1
$\frac{1}{3}$	R	1,-1	-1,1

- Is the mixed strategy profile an MSNE?
- To prove, one needs to show if there exists or does not exist a better mixed strategy for any of the players
- Expected utility for player 2 from L $= \frac{2}{3}(1) + \frac{1}{3}(-1) = 1/3$, and that from R $= -1/3$
- Expected utility will increase if player 2 moves probability mass from R to L
- $\implies$ this is not an MSNE
- Hints at some balance among the expected utilities at different strategies

	S\G	$\frac{1}{2}$ L	$\frac{1}{2}$ R
$\frac{1}{2}$	L	-1,1	1,-1
$\frac{1}{2}$	R	1,-1	-1,1

- Repeat the calculations

Finding an MSNE

- **support of a mixed strategy**

Definition (Support of a Mixed Strategy)

The support of a mixed strategy σ_i is the subset of the strategy space of i on which the mixed strategy σ_i has positive mass, and is denoted by

$$\delta(\sigma_i) = \{s_i \in S_i : \sigma_i(s_i) > 0\}.$$

- using the definition of support, we find a characterization of the MSNE

Theorem (Characterization of a MSNE)

A mixed strategy profile $(\sigma_i^*, \sigma_{-i}^*)$ *is a MSNE iff* $\forall i \in N$

1. $u_i(s_i, \sigma_{-i}^*)$ *is the same for all* $s_i \in \delta(\sigma_i^*)$, *and*
2. $u_i(s_i, \sigma_{-i}^*) \geq u_i(s_i', \sigma_{-i}^*), \ \forall \ s_i \in \delta(\sigma_i^*), s_i' \notin \delta(\sigma_i^*)$.

Application of the Characterization

Theorem (Characterization of a MSNE)

A mixed strategy profile $(\sigma_i^*, \sigma_{-i}^*)$ *is a MSNE iff* $\forall i \in N$

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- Penalty shootout game / matching coins game

1 \ 2	H	T
H	1,-1	-1,1
T	-1,1	1,-1

- Let the probability of player 1 choosing H be p – similarly, for player 2, let that probability be q
- **support** $= (\{\mathbf{H}\},\{\mathbf{H}\})$
  - for player 2, $s_2 = T$ violates condition 2
- **support** $= (\{\mathbf{H}\},\{\mathbf{H,T}\})$
  - the expected utility for player 2 has to be equal for H and T – cannot happen – violates condition 1

Application of Characterization (contd.)

- **support = ({H,T},{H,T})**
  - condition 2 is vacuously satisfied
  - consider player 1 chooses H w.p. p and player 2 chooses H w.p. q in the MSNE
  - condition 1 for player 1: $u_1(H,(q,1-q)) = u_1(T,(q,1-q))$, gives the value of q
  - condition 1 for player 2: $u_2((p,1-p),H) = u_2((p,1-p),T)$, gives the value of p
  - hence the MSNE for this game $((\frac{1}{2},\frac{1}{2}),(\frac{1}{2},\frac{1}{2}))$

Two more exercises (in class)

- Football or Cricket game

1\2	F	C
F	2,1	0,0
C	0,0	1,2

- An arbitrary added strategy for player 2

1 \ 2	F	C	D
F	2,1	0,0	1,1
C	0,0	1,2	2,0