CS711: Introduction to Game Theory and Mechanism Design

Teacher: Swaprava Nath

Elimination of Dominated Strategies, Two Player Zero Sum Games

Recap

- **dominance** cannot explain all reasonable outcomes

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- What happens to stability and security when certain strategies are eliminated (today)
- In general, utility at PSNE $\geqslant$ maxmin value (for each player)
- Situations where stability and security coincide (if time permits)

Iterated Elimination of Dominated Strategies

- Discussed this as a method to find equilibrium

1\2	L	C	R
T	1,2	2,3	0,3
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Consider an NFG $G = \langle N, (S_i)_{i \in N}, (u_i)_{i \in N} \rangle$, let $\widehat{s}_j \in S_j$ be a dominated strategy. Let $\widehat{G}$ be the residual game after removing the strategy $\widehat{s}_j$. The maxmin value of player j in $\widehat{G}$ is equal to her maxmin value in G.

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- non-triviality: to show that the conclusion holds even after taking the min over s_{-j} at the strategies of j
- formal proof as follows

Proof

- maxmin value of j in G

$$\underline{v}_j = \max_{s_j \in S_j} \min_{s_{-j} \in S_{-j}} u_j(s_j, s_{-j})$$

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Proof (contd.)

We have $\max_{s_j \in S_j \setminus \{\widehat{s}_j\}} \min_{s_{-j} \in S_{-j}} u_j(t_j, s_{-j}) \geqslant \min_{s_{-j} \in S_{-j}} u_j(\widehat{s}_j, s_{-j})$

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Exercise: elimination of dominated strategy of player j may increase the maxmin value of player $i \neq j$: find an example where it happens

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Equilibrium in a larger game $\implies$ equilibrium in a smaller game

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Theorem

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Proof: exercise.

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Answer: not if the eliminated strategies are dominated.

Theorem

Consider an NFG $G = \langle N, (S_i)_{i \in N}, (u_i)_{i \in N} \rangle$. Let $\widehat{s}_j \in S_j$ be a weakly dominated strategy of player $j \in N$. Let $\widehat{G}$ is generated from G by eliminating the strategy $\widehat{s}_j$. Every PSNE of $\widehat{G}$ is a PSNE of G.

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$$\implies u_j(s^*_j, s^*_{-j}) \geqslant u_j(t_j, s^*_{-j}) \geqslant u_j(\widehat{s}_j, s^*_{-j}), \quad \text{as } t_j \in \widehat{S}_j$$

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- The maxmin value is unaffected by the elimination of either strictly or weakly dominated strategies

Stability and Security Tension

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- The game is representable using just a matrix – **matrix games**

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- denote the matrix by u, the utility of Player 1

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  - (s_1^*, s_2^*) is a PSNE
- minimum utility and maximum loss of players 1 and 2:

$$\underline{v} := \max_{s_1 \in S_1} \min_{s_2 \in S_2} u(s_1, s_2) \quad \text{maxmin value}$$

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- Define

$$s_1^* \in \arg\max_{s_1 \in S_1} \min_{s_2 \in S_2} u(s_1, s_2), \text{ maxmin strategy for 1}$$

$$s_2^* \in \arg\min_{s_2 \in S_2} \max_{s_1 \in S_1} u(s_1, s_2), \text{ minmax strategy for 2}$$

A Characterization Theorem for Matrix Games

Theorem

A matrix game u has a PSNE (saddle point) if and only if $\underline{v} = \overline{v} = u(s_1^*, s_2^*)$, where s_1^* and s_2^* are maxmin strategy for player 1 and minmax strategy for 2 respectively.

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The last inequality holds by applying the same argument for player 2 and the fact that (s_1^*, s_2^*) is a PSNE $\implies u(s_1^*, s_2^*) \geqslant \overline{v} \geqslant \underline{v} \geqslant u(s_1^*, s_2^*)$
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$$\implies (s_1^*, s_2^*) \text{ is a PSNE}$$