CS711: Introduction to Game Theory and Mechanism Design

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Elimination of Dominated Strategies, Two Player Zero Sum Games

Recap

- **dominance** cannot explain all reasonable outcomes
- **pure strategy Nash equilibrium** – unilateral deviation, **stability**
- **max-min strategy** – another notion of rationality – *risk averse* players – **security**
- What happens to stability and security when certain strategies are eliminated (today)
- In general, utility at PSNE $\geqslant$ maxmin value (for each player)
- Situations where stability and security coincide (if time permits)

Iterated Elimination of Dominated Strategies

- Discussed this as a method to find equilibrium

1\2	L	C	R
T	1,2	2,3	0,3
M	2,2	2,1	3,2
B	2,1	0,0	1,0

- Order: T, R, B, C, Outcome: ML, Payoff: 2,2
- Order: B, L, C, T, Outcome: MR, Payoff: 3,2
- ...

Elimination of Dominated Strategies (contd.)

- Question: does it change the set of equilibria/maxmin value?
- On maxmin value – it is not affected for the player whose dominated strategy has been removed

Theorem

Consider an NFG $G = \langle N, (S_i)_{i \in N}, (u_i)_{i \in N} \rangle$, let $\widehat{s}_j \in S_j$ be a dominated strategy. Let $\widehat{G}$ be the residual game after removing the strategy $\widehat{s}_j$. The maxmin value of player j in $\widehat{G}$ is equal to her maxmin value in G.

- intuition: since $\widehat{s}_j$ is dominated, the utilities are no smaller in $\widehat{G}$ than G for j *at every* s_{-j} – so the max part is unaffected
- non-triviality: to show that the conclusion holds even after taking the min over s_{-j} at the strategies of j
- formal proof as follows

Proof

- maxmin value of j in G

$$\underline{v}_j = \max_{s_j \in S_j} \min_{s_{-j} \in S_{-j}} u_j(s_j, s_{-j})$$

- maxmin value of j in $\widehat{G}$

$$\widehat{\underline{v}}_j = \max_{s_j \in S_j \setminus \{\widehat{s}_j\}} \min_{s_{-j} \in S_{-j}} u_j(s_j, s_{-j})$$

- let t_j dominate $\widehat{s}_j$, clearly, $t_j \in S_j \setminus \{\widehat{s}_j\}$

$$u_j(t_j, s_{-j}) \geqslant u_j(\widehat{s}_j, s_{-j}), \ \forall s_{-j} \in S_{-j}$$

- Also

$$\min_{s_{-j} \in S_{-j}} u_j(t_j, s_{-j}) = u_j(t_j, \overline{s}_{-j}) \geqslant u_j(\widehat{s}_j, \overline{s}_{-j}) \geqslant \min_{s_{-j} \in S_{-j}} u_j(\widehat{s}_j, s_{-j})$$

$$\max_{s_j \in S_j \setminus \{\widehat{s}_j\}} \min_{s_{-j} \in S_{-j}} u_j(s_j, s_{-j}) \geqslant \min_{s_{-j} \in S_{-j}} u_j(t_j, s_{-j}) \geqslant \min_{s_{-j} \in S_{-j}} u_j(\widehat{s}_j, s_{-j})$$

Proof (contd.)

We have $\max_{s_j \in S_j \setminus \{\widehat{s}_j\}} \min_{s_{-j} \in S_{-j}} u_j(t_j, s_{-j}) \geqslant \min_{s_{-j} \in S_{-j}} u_j(\widehat{s}_j, s_{-j})$

Hence
$$\begin{aligned}\underline{v}_j &= \max_{s_j \in S_j} \min_{s_{-j} \in S_{-j}} u_j(s_j, s_{-j}) \\ &= \max \left\{ \max_{s_j \in S_j \setminus \{\widehat{s}_j\}} \min_{s_{-j} \in S_{-j}} u_j(s_j, s_{-j}), \min_{s_{-j} \in S_{-j}} u_j(\widehat{s}_j, s_{-j}) \right\} \\ &= \max_{s_j \in S_j \setminus \{\widehat{s}_j\}} \min_{s_{-j} \in S_{-j}} u_j(s_j, s_{-j}) = \widehat{\underline{v}}_j\end{aligned}$$

This completes the proof

Exercise: elimination of dominated strategy of player j may increase the maxmin value of player $i \neq j$: find an example where it happens

Preservation of Equilibria

Equilibrium in a larger game $\implies$ equilibrium in a smaller game

Theorem

Consider an NFG $G = \langle N, (S_i)_{i \in N}, (u_i)_{i \in N} \rangle$, and let $\widehat{G} = \langle N, (\widehat{S}_i)_{i \in N}, (u_i)_{i \in N} \rangle$ be the game derived from G through elimination of some strategies, i.e., $\widehat{S}_i \subseteq S_i, \ \forall i \in N$. If s^* is a PSNE in G, and if $s_i^* \in \widehat{S}_i$ for every $i \in N$, then s^* is an equilibrium in $\widehat{G}$.

Proof: exercise.

Question: can new equilibrium be generated due to elimination of strategies?
Answer: not if the eliminated strategies are dominated.

Theorem

Consider an NFG $G = \langle N, (S_i)_{i \in N}, (u_i)_{i \in N} \rangle$. Let $\widehat{s}_j \in S_j$ be a weakly dominated strategy of player $j \in N$. Let $\widehat{G}$ is generated from G by eliminating the strategy $\widehat{s}_j$. Every PSNE of $\widehat{G}$ is a PSNE of G.

Proof

- Strategy sets in $\widehat{G}$: $\widehat{S}_j := S_j \setminus \{\widehat{s}_j\}$ for j and $\widehat{S}_i := S_i$ for all $i \neq j$
- Let $s^* = (s^*_j, s^*_{-j})$ be a PSNE of $\widehat{G}$

$$u_i(s^*) \geqslant u_i(s_i, s^*_{-i}), \ \forall i \neq j, \forall s_i \in \widehat{S}_i = S_i$$
$$u_j(s^*) \geqslant u_j(s_j, s^*_{-j}), \ \forall s_j \in \widehat{S}_j$$

- to show that s^* is a PSNE of G, we need to show that there is no profitable deviation of any player in G
- for all $i \neq j$, this is clear from first set of inequalities
- for j, this is true for all the strategies except $\widehat{s}_j$
- need to show: no profitable deviation from s^*_j to $\widehat{s}_j$
- $\widehat{s}_j$ is a dominated strategy, hence $\exists t_j \neq \widehat{s}_j$ (hence $t_j \in \widehat{S}_j$)

$$u_j(t_j, s_{-j}) \geqslant u_j(\widehat{s}_j, s_{-j}), \ \forall s_{-j} \in S_{-j}$$
$$\implies u_j(t_j, s^*_{-j}) \geqslant u_j(\widehat{s}_j, s^*_{-j}), \quad \text{in particular}$$
$$\implies u_j(s^*_j, s^*_{-j}) \geqslant u_j(t_j, s^*_{-j}) \geqslant u_j(\widehat{s}_j, s^*_{-j}), \quad \text{as } t_j \in \widehat{S}_j$$

Summary of Elimination of Dominated Strategies

- Elimination of strictly dominated strategies have no effect on equilibria (exercise)
- Elimination of weakly dominated strategies may reduce the set of equilibria – but never adds new
- The maxmin value is unaffected by the elimination of either strictly or weakly dominated strategies

Stability and Security Tension

- Two player zero sum games
  - $\langle N, (S_i)_{i \in N}, (u_i)_{i \in N} \rangle$
  - $N = \{1, 2\}$, $u_1 + u_2 \equiv 0$
  - $u_1 = -u_2 = u$

Penalty shootout game

S\G	L	R
L	-1,1	1,-1
R	1,-1	-1,1

1\2	L	C	R
T	3,-3	-5,5	-2,2
M	1,-1	4,-4	1,-1
B	6,-6	-3,3	-5,5

- The game is representable using just a matrix – **matrix games**

Matrix Games

- Player 2's utilities are negative of the following numbers
- Player 2's maxmin strategy in the original game is the minmax strategy of this matrix

S\G	L	R	maxmin
L	-1	1	-1
R	1	-1	-1
minmax	1	1	

1\2	L	C	RR	maxmin
T	3	-5	-2	-5
MM	1	4	1	11
B	6	-3	-5	-5
minmax	6	4	11	

- What are the PSNEs of these games?

S\G	L	R
L	-1,1	1,-1
R	1,-1	-1,1

1\2	L	C	R
T	3,-3	-5,5	-2,2
M	1,-1	4,-4	1,-11,-1
B	6,-6	-3,3	-5,5

- denote the matrix by u, the utility of Player 1

PSNE and Saddle Point

- A saddle point: the value is maximum for player 1 and minimum for player 2
- PSNE?

Theorem

In a matrix game with utility matrix u, (s_1^*, s_2^*) is a saddle point if and only if it is a PSNE.

- Proof:
  - (s_1^*, s_2^*) is a saddle point $\Leftrightarrow$
  - $u(s_1^*, s_2^*) \geqslant u(s_1, s_2^*),\ \forall s_1 \in S_1$, and $u(s_1^*, s_2^*) \leqslant u(s_1^*, s_2),\ \forall s_2 \in S_2 \Leftrightarrow$
  - (s_1^*, s_2^*) is a PSNE
- minimum utility and maximum loss of players 1 and 2:

$$\underline{v} := \max_{s_1 \in S_1} \min_{s_2 \in S_2} u(s_1, s_2) \quad \text{maxmin value}$$

$$\overline{v} := \min_{s_2 \in S_2} \max_{s_1 \in S_1} u(s_1, s_2) \quad \text{minmax value}$$

Relation between maxmin and minmax

$$\underline{v} := \max_{s_1 \in S_1} \min_{s_2 \in S_2} u(s_1, s_2) \quad \text{maxmin value}$$

$$\overline{v} := \min_{s_2 \in S_2} \max_{s_1 \in S_1} u(s_1, s_2) \quad \text{minmax value}$$

Lemma

For two player zero sum games, $\underline{v} \leqslant \overline{v}$

Proof:

$$u(s_1, s_2) \geqslant \min_{t_2 \in S_2} u(s_1, t_2), \forall s_1, s_2 \text{ by definition of } \min$$

$$\max_{t_1 \in S_1} u(t_1, s_2) \geqslant \max_{t_1 \in S_1} \min_{t_2 \in S_2} u(t_1, t_2), \forall s_2$$

$$\min_{t_2 \in S_2} \max_{t_1 \in S_1} u(t_1, t_2) \geqslant \max_{t_1 \in S_1} \min_{t_2 \in S_2} u(t_1, t_2)$$

Maxmin and Minmax with PSNE

S\G	L	R	maxmin
L	-1	1	-1
R	1	-1	-1
minmax	1	1	

$\underline{v} = -1 < 1 = \overline{v}$

1\2	L	C	RR	maxmin
T	3	-5	-2	-5
MM	1	4	1	11
B	6	-3	-5	-5
minmax	6	4	11	

$\underline{v} = 1 = \overline{v}$

- Seems like existence of PSNE and the values have a connection
- Define

$$s_1^* \in \arg\max_{s_1 \in S_1} \min_{s_2 \in S_2} u(s_1, s_2), \text{ maxmin strategy for 1}$$

$$s_2^* \in \arg\min_{s_2 \in S_2} \max_{s_1 \in S_1} u(s_1, s_2), \text{ minmax strategy for 2}$$

A Characterization Theorem for Matrix Games

Theorem

A matrix game u has a PSNE (saddle point) if and only if $\underline{v} = \overline{v} = u(s_1^*, s_2^*)$, where s_1^* and s_2^* are maxmin strategy for player 1 and minmax strategy for 2 respectively.

Proof: $(\Rightarrow)$ given: u has a PSNE, say (s_1^*, s_2^*)

$$\begin{aligned} & u(s_1^*, s_2^*) \geqslant u(s_1, s_2^*), \forall s_1 \in S_1 \\ \implies & u(s_1^*, s_2^*) \geqslant \max_{t_1 \in S_1} u(t_1, s_2^*) \\ & \geqslant \min_{t_2 \in S_2} \max_{t_1 \in S_1} u(t_1, t_2) = \overline{v} \\ & \geqslant \underline{v} = \max_{t_1 \in S_1} \min_{t_2 \in S_2} u(t_1, t_2) \geqslant u(s_1^*, s_2^*) \end{aligned}$$

The last inequality holds by applying the same argument for player 2 and the fact that (s_1^*, s_2^*) is a PSNE $\implies u(s_1^*, s_2^*) \geqslant \overline{v} \geqslant \underline{v} \geqslant u(s_1^*, s_2^*)$
$\implies u(s_1^*, s_2^*) = \overline{v} = \underline{v}$
this and PSNE implies s_1^* and s_2^* are maxmin strategy for player 1 and minmax strategy for 2 respectively. (needs proof, exercise)

Characterization Theorem (contd.)

Proof (contd.):
($\Leftarrow$) given: $u(s_1^*, s_2^*) = \overline{v} = \underline{v} = v$ (say), where s_1^* and s_2^* are maxmin strategy for player 1 and minmax strategy for 2 respectively.

$$\begin{aligned} u(s_1^*, s_2) &\geqslant \min_{t_2 \in S_2} u(s_1^*, t_2), \forall s_2 \in S_2, \text{ by defn of } \min \\ &= \max_{t_1 \in S_1} \min_{t_2 \in S_2} u(t_1, t_2), \text{ by defn of } s_1^*, \forall s_2 \in S_2 \\ &= v, \forall s_2 \in S_2 \end{aligned}$$

similarly we can show that $u(s_1, s_2^*) \leqslant v, \forall s_1 \in S_1$

$$\implies u(s_1^*, s_2^*) = v$$

$$\implies (s_1^*, s_2^*) \text{ is a PSNE}$$