# CS711: Introduction to Game Theory and Mechanism Design

**Teacher: Swaprava Nath**

Domination, Elimination of Dominated Strategies, Nash Equilibrium

# Domination

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**Definition (Dominated Strategy)**

A strategy $s_i' \in S_i$ of player $i$ is **strictly dominated** if there exists another strategy $s_i$ of $i$ such that for every strategy profile $s_{-i} \in S_{-i}$ of the other players

$$u_i(s_i, s_{-i}) > u_i(s_i', s_{-i}).$$

A strategy $s_i' \in S_i$ of player $i$ is **weakly dominated** if there exists another strategy $s_i$ of $i$ such that for every strategy profile $s_{-i} \in S_{-i}$ of the other players

$$u_i(s_i, s_{-i}) \geqslant u_i(s_i', s_{-i}),$$

and there exists some $\tilde{s}_{-i} \in S_{-i}$ such that

$$u_i(s_i, \tilde{s}_{-i}) > u_i(s_i', \tilde{s}_{-i}).$$

# Domination (Contd.)

**Definition (Dominant Strategy)**

A strategy $s_i$ is **strictly (weakly) dominant strategy** for player $i$ if $s_i$ strictly (weakly) dominates all other $s_i' \in S_i \setminus \{s_i\}$.

**Definition (Dominant Strategy Equilibrium)**

A strategy profile $(s_i^*, s_{-i}^*)$ is a **strictly (weakly) dominant strategy equilibrium (SDSE (WDSE))** if $s_i^*$ is a strictly (weakly) dominant strategy for every $i, i \in N$.

# Examples

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- **Refine the equilibrium concept**

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**Definition (Pure strategy Nash equilibrium)**

A strategy profile $(s_i^*, s_{-i}^*)$ is a *pure strategy Nash equilibrium* (PSNE) if $\forall i \in N$ and $\forall s_i \in S_i$

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## Definition (Best response set)

A best response of agent $i$ against the strategy profile $s_{-i}$ of the other players is a strategy that gives the maximum utility against the $s_{-i}$ chosen by other players, i.e.,

$$B_i(s_{-i}) \; = \; \{s_i \in S_i : u_i(s_i, s_{-i}) \geqslant u_i(s_i', s_{-i}), \; \forall s_i' \in S_i\}.$$

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- multiplicity of equilibria – which one should players coordinate to

# Risk averse players

- **risky** equilibrium

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# Max-min and Dominance

Relationship of max-min strategies and dominant strategies

**Theorem**

*If $s_i^*$ is a dominant strategy for player $i$, then it is a max-min strategy for player $i$ as well, for all $i \in N$. Such a strategy is a best response of player $i$ to* **any** *strategy profile of the other players.*

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$$s_i^* \in \arg\max_{s_i \in S_i} \min_{s_{-i} \in S_{-i}} u_i(s_i, s_{-i})$$

- The second part of the theorem follows from definition

∎

# Max-min and Dominance

Relationship of max-min strategies and dominant strategies

**Theorem**

*If $s_i^*$ is a dominant strategy for player $i$, then it is a max-min strategy for player $i$ as well, for all $i \in N$. Such a strategy is a best response of player $i$ to* **any** *strategy profile of the other players.*

**Proof sketch:** [for strictly dominant strategies]

- Let $s_i^*$ is the strictly dominant strategy of player $i$

$$u_i(s_i^*, s_{-i}) > u_i(s_i', s_{-i}), \; \forall s_i' \in S_i \setminus \{s_i^*\}, s_{-i} \in S_{-i}$$

- holds for every $\bar{s}_{-i}^{s_i'} \in \arg\min_{s_{-i} \in S_{-i}} u_i(s_i', s_{-i})$
- hence

$$u_i(s_i^*, \bar{s}_{-i}^{s_i'}) > u_i(s_i', \bar{s}_{-i}^{s_i'}), \; \forall s_i' \in S_i \setminus \{s_i^*\}$$

$$s_i^* \in \arg\max_{s_i \in S_i} \min_{s_{-i} \in S_{-i}} u_i(s_i, s_{-i})$$

- The second part of the theorem follows from definition

■

**Exercise:** finish the proof for weakly dominant strategies

# More results

**Theorem**

*If every player $i \in N$ has a* strictly dominant *strategy $s_i^*$, then the strategy profile $(s_1^*, \ldots, s_n^*)$ is the unique equilibrium point of the game and also the unique profile of max-min strategies.*

**Proof:** exercise (can use the previous result)

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Relationship with pure strategy Nash equilibrium

**Theorem**

*For every PSNE $s^* = (s_1^*, \dots, s_n^*)$ of a normal form game satisfies $u_i(s^*) \geqslant \underline{v}_i$, for all $i \in N$.*

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| **1\2** | L | R |
|---|---|---|
| T | 2,1 | 2,-20 |
| M | 3,0 | -10,1 |
| B | -100,2 | 3,3 |

# Proof

**Proof:**

$$u_i(s_i, s^*_{-i}) \geqslant \min_{s_{-i} \in S_{-i}} u_i(s_i, s_{-i}), \ \forall s_i \in S_i, \text{ by definition of } \min$$

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$$\text{Now, } u_i(s^*_i, s^*_{-i}) \geqslant u_i(s_i, s^*_{-i}),\ \forall s_i \in S_i,\ \text{by the best response definition}$$

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$$u_i(s_i, s^*_{-i}) \geqslant \min_{s_{-i} \in S_{-i}} u_i(s_i, s_{-i}), \ \forall s_i \in S_i, \text{ by definition of } \min$$

$$\text{Now, } u_i(s^*_i, s^*_{-i}) \geqslant u_i(s_i, s^*_{-i}), \ \forall s_i \in S_i, \text{ by the best response definition}$$

$$\text{Hence, } u_i(s^*_i, s^*_{-i}) = \max_{s_i \in S_i} u_i(s_i, s^*_{-i}) \geqslant \max_{s_i \in S_i} \min_{s_{-i} \in S_{-i}} u_i(s_i, s_{-i}) = \underline{v}_i$$

■