CS711: Introduction to Game Theory and Mechanism Design

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Domination, Elimination of Dominated Strategies, Nash Equilibrium

Domination

- Normal form game $\langle N, (S_i)_{i \in N}, (u_i)_{i \in N} \rangle$

Definition (Dominated Strategy)

A strategy $s'_i \in S_i$ of player i is **strictly dominated** if there exists another strategy s_i of i such that for every strategy profile $s_{-i} \in S_{-i}$ of the other players

$$u_i(s_i, s_{-i}) > u_i(s'_i, s_{-i}).$$

A strategy $s'_i \in S_i$ of player i is **weakly dominated** if there exists another strategy s_i of i such that for every strategy profile $s_{-i} \in S_{-i}$ of the other players

$$u_i(s_i, s_{-i}) \geqslant u_i(s'_i, s_{-i}),$$

and there exists some $\tilde{s}_{-i} \in S_{-i}$ such that

$$u_i(s_i, \tilde{s}_{-i}) > u_i(s'_i, \tilde{s}_{-i}).$$

Domination (Contd.)

Definition (Dominant Strategy)

A strategy s_i is **strictly (weakly) dominant strategy** for player i if s_i strictly (weakly) dominates all other $s'_i \in S_i \setminus \{s_i\}$.

Definition (Dominant Strategy Equilibrium)

A strategy profile (s^*_i, s^*_{-i}) is a **strictly (weakly) dominant strategy equilibrium (SDSE (WDSE))** if s^*_i is a strictly (weakly) dominant strategy for every $i, i \in N$.

Examples

- Neighboring kingdoms' dilemma

A**B**	Agriculture	Defense
Agriculture	5,5	0,6
Defense	6,0	1,1

- Do the players have a dominant strategy? Which kind?

Examples (contd.)

- One indivisible object for sale
- Two players, each having a *value* for the object, v_1 and v_2
- Rules:
  - each player can choose a number in $[0, M]$ which reflects her willingness to buy the object
  - player quoting the larger number (tie broken in favor of player 1) wins
  - pays the losing players chosen number
  - utility of the winning player is $=$ her value - her payment
  - utility of the losing player is zero
- What is the normal form representation of this game?
- $N = \{1, 2\}, S_1 = S_2 = [0, M]$

$$u_1(s_1, s_2) = \begin{cases} v_1 - s_2 & \text{if } s_1 \geqslant s_2 \\ 0 & \text{otherwise} \end{cases}; \qquad u_2(s_1, s_2) = \begin{cases} v_2 - s_1 & \text{if } s_1 < s_2 \\ 0 & \text{otherwise} \end{cases}$$

- Do the players have a dominant strategy? Which kind?

Iterated Elimination of Dominated Strategies

- Rational players do not play dominated strategies
- To find a reasonable outcome, eliminate the dominated strategies
- For strictly dominated strategies, the order of elimination does not matter – always reaches the same residual game
- For weakly dominated strategies, the order matters
- It can also eliminate some reasonable outcomes

1\2	L	C	R
T	1,2	2,3	0,3
M	2,2	2,1	3,2
B	2,1	0,0	1,0

- Order: T, R, B, C, Outcome: ML, Payoff: 2,2
- Order: B, L, C, T, Outcome: MR, Payoff: 3,2
- ...

DSE: Does it always exist?

- Coordination game: drive to the left of right?

1\2	L	R
L	1,1	0,0
R	0,0	1,1

- Football or Cricket game

1\2	F	C
F	2,1	0,0
C	0,0	1,2

- Dominance cannot explain a reasonable outcome in this game – then what?
- **Refine the equilibrium concept**

Nash Equilibrium

- This is a strategy profile from where no player gains by a unilateral deviation

Definition (Pure strategy Nash equilibrium)

A strategy profile (s_i^*, s_{-i}^*) is a *pure strategy Nash equilibrium* (PSNE) if $\forall i \in N$ and $\forall s_i \in S_i$

$$u_i(s_i^*, s_{-i}^*) \geqslant u_i(s_i, s_{-i}^*).$$

1\2	F	C
F	2,1	0,0
C	0,0	1,2

- A best response view

Definition (Best response set)

A best response of agent i against the strategy profile s_{-i} of the other players is a strategy that gives the maximum utility against the s_{-i} chosen by other players, i.e.,

$$B_i(s_{-i}) \ = \ \{s_i \in S_i : u_i(s_i, s_{-i}) \geqslant u_i(s_i', s_{-i}), \ \forall s_i' \in S_i\}.$$

PSNE: Best Response View

Definition (PSNE)

A strategy profile (s_i^*, s_{-i}^*) is a *pure strategy Nash equilibrium* if $\forall i \in N, s_i^* \in B_i(s_{-i}^*)$.

Properties of PSNE

- Stability – a point from where no player would like to deviate
- Self-enforcing agreement among the players

But,

- this still assumes players to be **fully rational**
- multiplicity of equilibria – which one should players coordinate to

Risk averse players

- **risky** equilibrium

1\2	L	R
T	2,1	2,-20
M	3,0	-10,1
B	-100,2	3,3

- player 1 may choose T, since that is least risky
- player 2 should then choose L
- another aspect of rationality – where players make *pessimistic* estimate about others' play (instead of utility maximization)
- this **worst case optimal** choice is known as max-min strategy

$$s_i^* \in \arg\max_{s_i \in S_i} \min_{s_{-i} \in S_{-i}} u_i(s_i, s_{-i})$$

maxmin value

$$\underline{v}_i = \max_{s_i \in S_i} \min_{s_{-i} \in S_{-i}} u_i(s_i, s_{-i})$$

$$u_i(s_i^*, t_{-i}) \geqslant \underline{v}_i, \ \forall t_{-i} \in S_{-i}$$

Max-min and Dominance

Relationship of max-min strategies and dominant strategies

Theorem

If s_i^* is a dominant strategy for player i, then it is a max-min strategy for player i as well, for all $i \in N$. Such a strategy is a best response of player i to **any** *strategy profile of the other players.*

Proof sketch: [for strictly dominant strategies]

- Let s_i^* is the strictly dominant strategy of player i

$$u_i(s_i^*, s_{-i}) > u_i(s_i', s_{-i}), \ \forall s_i' \in S_i \setminus \{s_i^*\}, s_{-i} \in S_{-i}$$

- holds for every $\overline{s}_{-i}^{s_i'} \in \arg\min_{s_{-i} \in S_{-i}} u_i(s_i', s_{-i})$
- hence

$$u_i(s_i^*, \overline{s}_{-i}^{s_i'}) > u_i(s_i', \overline{s}_{-i}^{s_i'}), \ \forall s_i' \in S_i \setminus \{s_i^*\}$$

$$s_i^* \in \arg\max_{s_i \in S_i} \min_{s_{-i} \in S_{-i}} u_i(s_i, s_{-i})$$

- The second part of the theorem follows from definition

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Exercise: finish the proof for weakly dominant strategies

More results

Theorem

If every player $i \in N$ has a strictly dominant *strategy s_i^*, then the strategy profile $(s_1^*, \ldots, s_n^*)$ is the unique equilibrium point of the game and also the unique profile of max-min strategies.*

Proof: exercise (can use the previous result)

Relationship with pure strategy Nash equilibrium

Theorem

For every PSNE $s^* = (s_1^*, \ldots, s_n^*)$ of a normal form game satisfies $u_i(s^*) \geqslant \underline{v}_i$, for all $i \in N$.

1\2	L	R
T	2,1	2,-20
M	3,0	-10,1
B	-100,2	3,3

Proof

Proof:

$$u_i(s_i, s^*_{-i}) \geqslant \min_{s_{-i} \in S_{-i}} u_i(s_i, s_{-i}),\ \forall s_i \in S_i,\ \text{by definition of } \min$$

$$\text{Now, } u_i(s^*_i, s^*_{-i}) \geqslant u_i(s_i, s^*_{-i}),\ \forall s_i \in S_i,\ \text{by the best response definition}$$

$$\text{Hence, } u_i(s^*_i, s^*_{-i}) = \max_{s_i \in S_i} u_i(s_i, s^*_{-i}) \geqslant \max_{s_i \in S_i} \min_{s_{-i} \in S_{-i}} u_i(s_i, s_{-i}) = \underline{v}_i$$

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