CS-698W: Game Theory and Collective Choice	**Jul-Nov 2017**

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Disclaimer: *These notes aggregate content from several texts and have not been subjected to the usual scrutiny deserved by formal publications. If you find errors, please bring to the notice of the Instructor at swaprava.cse.iitk.ac.in .*

30.1 Recap

In the previous lecture we studied that VCG mechanism is individually rational. We also looked at internet advertising and position auctions to sell multiple ads on a webpage.
We also studied the winner determination problem which is restated for the purpose of this lecture.

Definition 30.1 (winner determination problem) *The optimal allocation x^* should be determined such that the sum of overall reported value of all players is maximised.*

$$x^* = \underset{x}{\operatorname{argmax}} \sum_{i \in N} \hat{v}_i(x)$$

30.2 Winner Determination Problem (continued)

Lets look at the following theorem regarding the winner determination problem.

Theorem 30.2 *If an allocation solves the Winner determination problem, then it must be a rank-by-revenue mechanism.*

Proof: We prove this theorem using contradiction.Assume for contradiction that allocation x is optimal and bids of 1 and 2 are such that,$eCTR_1.b_1 > eCTR_@.b_2$
Also, 2 is placed above 1. So, WLOG $x_2 = 1$ and $x_1 = 2$. Consider a different allocation x' with all the agents except 1 and 2 getting the same position. In this allocation $x'_1 = 1$ and $x'_2 = 2$.Now.

$$\sum_{i \in N} \hat{v}_i(x') = pos_1(eCTR_1.b_1) + pos_2(eCTR_2.b_2) + \sum_{k \neq 1,2} pos_{x_k}(eCTR_k.b_k) \tag{30.1}$$

$$\sum_{i \in N} \hat{v}_i(x) = pos_2(eCTR_1.b_1) + pos_1(eCTR_2.b_2) + \sum_{k \neq 1,2} pos_{x_k}(eCTR_k.b_k) \tag{30.2}$$

Equation (30.1) - (30.2)
$\sum_{i \in N} \hat{v}_i(x') - \sum_{i \in N} \hat{v}_i(x) = (pos_1 - pos_2)(eCTR_1.b_1 - eCTR_2.b_2)$
As both terms in the RHS are non-negative, $\sum_{i \in N} \hat{v}_i(x') - \sum_{i \in N} \hat{v}_i(x) > 0$

This is a contradiction to x being optimal. This concludes the proof of the theorem. ■
Note An advantage to be noted here is that the winner determination problem is poly-time.

30.3 VCG in Position Auction

We have decided an allocation mechanism. We now need payments to implement it in DSIC. The natural candidate for this is VCG payment. VCG payment is used by twitter, facebook for ads.

Given bids in $(b_1, b_2..., b_n)$which are ordered Without loss of generality suct that, $eCTR_1.b_1 \geq eCTR_2.b_2 \geq .... \geq eCTR_n.b_n$.
The allocation x^* is such that $x^*_{-i} = i$ for $i = 1, 2...n$. The payment used is VCG.
Lets define $x^*_{-i} \in \text{argmax}_x \sum_{j \neq i} \hat{v}_i(x)$. Now the payment of agent i in VCG is given by,

$$p_i^{VCG}(b) = \sum_{j \neq i} \hat{v}_i(x^*_{-i}) + \sum_{j \neq i} \hat{v}_i(x^*)$$

$$= \sum_{j=i}^{n-1} pos_j(eCTR_{j+1}.b_{j+1}) - \sum_{j=i}^{n-1} pos_{j+1}(eCTR_{j+1}.b_{j+1})$$

So

$$p_i^{VCG}(b) = \begin{cases} \sum_{j=i}^{n-1} (pos_j - pos_{j+1})(eCTR_{j+1}.b_{j+1}) & i = 1, 2...n-1 \\ 0 & i = n \end{cases}$$

This gives us the Total expected payment. To convert to payment per click, we need to normalize with $pos_i.eCTR_i$. So payment per click $= \frac{1}{pos_i.eCTR_i} p_i^{VCG}(b)$

Observation: Allocation wrt. the rank revenue and payment wrt. VCG is DSIC if the eCTR's are accurate.

30.4 Comparision of VCG and GSP in position auction

Generalized Second Price auction is used by Google,Bing etc which charges every agent the second highest expected bid. This payment method is simple and easy to explain. It also has similarities with second price auction but also has serious flaws/limitations.Consider the following example

revenue before position effect	v_i	$eCTR_iG$	$(pos_j$
2	10	0.2	1
4	8	0.5	0.2
4.2	6	0.7	0.1

Both VCG and GSP allocate the slots to solve the winner determination problem (i.e. rank-by-revenue allocation) with slot1 going to player 3, slot 2 to player 2 and slot 3 to player1.
For VCG, payment of slot 1 (for player 3) is $p_i^{VCG}(b) = \sum_{j=i}^{n-1} (p_j - p_{j+1})(eCTR_{j+1}.b_{j+1}) = 0.8*4 + 0.1*2 = 3.4$.
Similarly payment for 2nd slot$= 0.1 * 2 = 0.2$. Payment for slot 3 is equal to 0.

Utility of slot $1 = 4.2 * 1 - 3.4 = 0.8$
Utility of slot $2 = 4 * 0.2 - 0.2 = 0.6$
Utility of slot $3 = 0.1 * 2 + 0 = 0.2$
For GSP,
Payment for slot 1 (player 3) $= 4 * 0.2 = 0.8$
Payment for slot 2 (player 2) $= 2 * 0.1 = 0.2$
Utility for Slot $1 = 4.2 * 1 - 0.8 = 3.4$
Utility for Slot $2 = 4 * 0.2 - 0.2 = 0.6$

An interesting case to consider is if player 2 overbids. Say player 2 bids 8.6 to change the allocation.
So $b_2 = 8.6$. In this case, the payment under VCG becomes $= 0.8 * 4.2 + 0.1 * 2 = 3.56$
The utility of player 2(now slot 1) $= 4 - 3.56 = 0.44$. We see that utility has decreased for player 2. This is because VCG is DSIC.
Under GSP, payment of player $2 = 4.2 * 0.2 = 0.84$
Utility of player $2 = 4 * 1 - 0.84$. Here utility increases. This is because GSP is NOT DSIC (or not truthful).

30.5 Desirable properties of VCG

1. VCG is DSIC. Hence there is very low cognitive load on bidders
2. VCG never runs into deficits (in some settings). It charges the marginal contribution to the other agents as payments and gets its own marginal contribution as payoff.
3. VCG never charges a losing agent
4. It is Individually rational for agent to participate.