CS-698W: Game Theory and Collective Choice **Jul-Nov 2017**

Lecture 28: Game Theory – An Introduction

Lecturer: Swaprava Nath *Scribe(s): Dhawal Upadhyay*

Disclaimer: *These notes aggregate content from several texts and have not been subjected to the usual scrutiny deserved by formal publications. If you find errors, please bring to the notice of the Instructor.*

28.1 Revisiting VCG

In VCG mechanism, we defined
$f^{AE}(\theta) \ \in \ \underset{a \in A}{argmax} \sum_{i \in N} v_i(a, \theta_i)$
$p_i^{VCG}(\theta) = \underset{b \in A}{max} \sum_{j \neq i} v_j(b, \theta_j) \ - \ \sum_{j \neq i} v_j(f^{AE}(\theta), \theta_j)$

Here $f^{AE}(\theta)$ is the allocation that maximises total utility of all players. Payment to be made by agent i is given by p_i.
Let's look at the utility of player i

$$\begin{aligned} v_i(\theta_i, \theta_{-i}|\theta_i) &= v_i(f^{AE}(\theta), \theta_i) - p_i^{VCG}(\theta) \\ &= v_i(f^{AE}(\theta), \theta_i) - \underset{b \in A}{max} \sum_{j \neq i} v_j(b, \theta_j) \ + \ \sum_{j \neq i} v_j(f^{AE}(\theta), \theta_j) \\ &= \sum_{j \in N} v_j(f^{AE}(\theta), \theta_j) - \underset{b \in A}{max} \sum_{j \neq i} v_j(b, \theta_j) \end{aligned}$$

The first term is the max total social utility. The second term is the max total utility in is absence. The difference gives us the marginal contribution of i in the social welfare. This is another way of interpreting the utility in VCG mechanism.

28.2 Illustration of VCG payments

- **Single Object Allocation**: Consider agent $i \in N$. Every agent has some value when the object is assigned to him, and zero otherwise. *Efficiency* requires the object to go to the agent who values it the most. So if is value/bid is maximum, he is assigned the object.
  payment$_i = \underset{a \in A}{max} \sum_{j \neq i} v_j(b, \theta_j) \ - \ \sum_{j \neq i} v_j(f^{AE}(\theta), \theta_j)$
  The first term is the second highest bid. The second term becomes zero if i's bid is highest. Thus the payment made by the highest bidder is equal to the second highest bid. This is the second price auction we have seen many times. So, in the case of single object allocation, VCG mechanism is equivalent to second price auction.
- **Public Project Allocation** In this case, the first term of payment function is the social welfare of all agents except agent i if i wasn't present. The second term is the social welfare of other agents in i's presence. Hence, another interpretation of the payment: Loss in social welfare of other agents because of agent i's presence.

Let's look at an example:

	Football	Library	Museum
A	0	70	50
B	95	10	50
C	10	50	50

$A = F, L, M$
Efficient VCG allocation: M
A pays: 105 - 100 = 5
Without A's presence, best allocation would've been F, with total utility of other agents = 105. In A's presence, utility of other agents = 50+50 = 100. Difference = 5.
Similarly, you can verify-
B pays: 120 - 100 = 20
C pays: 100 - 100 = 0

Observation: payment values of VCG are always positive (irrespective of utility function)- first term is always at least as large as second term.

In the above example, only those agents are charged whose presence changes the outcome. These agents are called *pivotal* agents.

- **Combinatorial Allocation**: Sale of multiple objects.

	ϕ	{1}	{2}	{1,2}
v_1	0	8	6	12
v_2	0	9	4	14

Efficient allocation - {1} goes to player 2, $\{2\} \rightarrow 1$
Here, type is the value itself, so we can also write $v_i(a, \theta_i) = \theta_i(a)$
$p_1^{VCG}(\theta_1, \theta_2) = \underset{a \in A}{max} \sum_{j \neq 1} \theta_j(a) \ - \ \sum_{j \neq 1} \theta_j(f(v))$
= 14 - 9 = 5; Payoff = 6 - 5 = 1
$p_2^{VCG}(\theta_1, \theta_2)$ = 12 - 6 = 6; Payoff = 9 - 6 = 3

28.3 VCG mechanism in combinatorial auction

VCG ha several useful properties:
1) It is DSIC.
2) It is efficient.
3) Payments are non-negative (hence no subsidy).

Combinatorial Auction notaion: $M = \{1, 2 \cdots m\}$ set of objects
Set of bundles $\Omega = \{S : S \subseteq M\}$
Type of agent i is $\theta_i : \ \Omega \rightarrow R$
Hence $\theta_i(S)$, $S \in \Omega$ is the value of agent i for bundle S. We assume $\theta_i(S) \geq 0 \ \forall S \in \Omega$
An allocation of objects is given by $X = \{X_0, X_1 \cdots X_n\}$, $X_i \in \Omega$, $X_i \cap X_j = \phi$ if $i \neq j$, and $\cup_{i=0}^{N} X_i = M$
A is the collection of such Xs.
X_0 is the set of uncallocated objects, X_i is the bundle allocated to i. Assume $\theta_i(\phi) = 0$
Note: The model is such that the valuations have no externalities (selfish valuations), ie $v_i(X, \theta_i) = \theta_i(X_i)$

(does not depend on allocation of other agents).

Claim 1: The payment for an agent who gets no object in the allocation is zero.
Proof: Say agent i gets no object in the efficient allocation, ie $X \in \underset{x \in A}{argmax} \sum_{i \in N} v_i(x, \theta_i)$, and $X_i = \phi$
VCG payment's first term considers allocation excluding agent i - $Y \in \underset{y \in A}{argmax} \sum_{j \neq i} v_j(y, \theta_j)$
We have seen VCG payment is always non-negative , ie $p_i^{VCG} \geq 0$ [no-subsidy]

$p_i^{VCG} = \sum_{j \neq i} v_j(Y, \theta_j) - \sum_{j \neq i} v_j(X, \theta_j)$

Adding $v_i(Y, \theta_i) \geq 0$ and subtracting $v_i(X, \theta_i) = 0$

$p_i^{VCG} \leq \sum_{j \in N} v_j(Y, \theta_j) - \sum_{j \in N} v_j(X, \theta_j)$
≤ 0

Hence $p_i^{VCG} = 0$