CS698W: Game Theory and Collective Choice **Jul-Nov 2017**

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27.1 Pareto Optimality

Definition 27.1.1 *A direct mechanism $(f, (p_1, p_2, \ldots, p_n))$ is pareto optimal (PO) if at every type profile $\theta \in \Theta$, $\nexists$ an allocation $b \in A$ and a payment vector $(\pi_1, \pi_2 \ldots, \pi_n)$ with $\sum_{i \in N} \pi_i \geqslant \sum_{i \in N} p_i(\theta)$ such that,*

$$\begin{aligned} v_i(b, \theta_i) - \pi_i &\geqslant v_i(f(\theta), \theta_i) - p_i(\theta) \qquad \text{for all } i \in N, \text{ and,} \\ v_j(b, \theta_j) - \pi_j &> v_j(f(\theta), \theta_j) - p_j(\theta) \qquad \text{for some } j \in N. \end{aligned}$$

Hence for a Pareto optimal mechanism, agents' payoffs are maximal at every type profile. Improving the payoff of one agent will result in the reduction of the payoff some other agent.

27.2 Relation between Pareto Optimality and Allocative Efficiency in Quasi-linear Domain

Recall that an allocation function is allocatively efficient (AE) if it maximizes the social welfare, i.e.,

$$f^{AE}(\theta) \in \arg\max_{a \in A} \sum_{i \in N} v_i(a, \theta_i)$$

Theorem 27.1 *A mechanism (f, p) is Pareto optimal iff it is allocatively efficient.*

Proof: $(\Rightarrow)$ We first prove that if a mechanism is pareto optimal then it is AE. To do so we show that !AE $\implies$!PO. Since f is not AE $\exists\, b \in A$ s.t .

$$\sum_{i \in N} v_i(b, \theta_i) > \sum_{i \in N} v_i(f(\theta), \theta_i), \text{ for some } \theta \in \Theta.$$

Define

$$\delta := \sum_{i \in N} v_i(b, \theta_i) - \sum_{i \in N} v_i(f(\theta), \theta_i).$$

By definition, $\delta > 0$. Now define a payment for every $i \in N$

$$\begin{aligned} &\pi_i = v_i(b, \theta_i) - v_i(f(\theta), \theta_i) + p_i(\theta) - \delta/n \\ \Rightarrow \quad &(v_i(b, \theta_i) - \pi_i) - (v_i(f(\theta), \theta_i) - p_i(\theta)) = \delta/n > 0 \end{aligned}$$

Hence the new allocation and payment yields more payoff to every agent. Moreover, we get $\sum_{i\in N}\pi_i = \sum_{i\in N} p_i(\theta)$. Hence, (f,p) is not PO.

($\Leftarrow$) To prove the converse, we show that !PO $\implies$!AE. If (f,p) is not PO, $\exists\, b, \pi, \theta$ such that the following holds.

$$\sum_{i\in N}\pi_i \geqslant \sum_{i\in N} p_i(\theta)$$
$$v_i(b,\theta_i) - \pi_i \geqslant v_(f(\theta),\theta_i) - p_i(\theta), \forall\ i\in N, \text{ and,}$$
$$v_j(b,\theta_j) - \pi_j > v_j(f(\theta),\theta_j) - p_j(\theta), \text{ for some } j\in N.$$

Summing over the last two inequalities over all $i\in N$, we get

$$\sum_{i\in N} v_i(b,\theta_i) - \sum_{i\in N}\pi_i > \sum_{i\in N} v_i(f(\theta),\theta_i) - \sum_{i\in N} p_i$$
$$\Rightarrow \quad \sum_{i\in N} v_i(b,\theta_i) - \sum_{i\in N} v_i(f(\theta),\theta_i) > \sum_{i\in N}\pi_i - \sum_{i\in N} p_i \geqslant 0.$$

Which proves that f is not AE. ■

27.3 Implementability of Allocation Rules

We call an allocation rule $f:\Theta\to A$ *implementable* if $\exists\, p$ such that (f,p) is DSIC. We show that the efficient rule is implementable. It is implemented by a class of payments known as the Groves class of payments.

Definition 27.3.1 (Groves class of payments) *Let $h_i : \Theta_{-i}\to\mathbb{R}$ be an arbitrary function for every $i\in N$. The Groves class of payments is defined as the payment rules defined as*

$$p_i{}^G(\theta) = h_i(\theta_{-i}) - \sum_{j\neq i} v_j(f^{AE}(\theta),\theta_j).$$

The mechanism (f^{AE}, p^G) is called the *Groves class of mechanisms.*

Definition 27.3.1 *Consider four agents and one indivisible item is to be allocated. Value of the agents are* 10*,* 8*,* 6 *and* 4 *respectively when they receive the item and zero otherwise. Suppose*

$$h_i(\theta_{-i}) = 10 \textit{ for all } \theta_{-i} \textit{ for all } i\in N.$$

Clearly, the efficient allocation is to give the item to agent 1. Since the h_i function is a constant, and the second term of the Groves payment is zero for agent 1 and 10 for every other agent, the payments are 10 *,* 0 *,* 0 *and* 0 *respectively. Therefore this payment rule charges 10 to the winning agent and zero to others.*

However, this class also admits very surprising payments. For example, if we consider $h_i(\theta_{-i}) = \sum_{j\neq i}\frac{\theta_j}{2}$, one can find that the Groves payments will be $9, 0, 1, 2$ respectively. These are surprising since the agents who do not receive the item are also asked to pay. Though the payments are surprising enough, they satisfy one very important property.

Theorem 27.2 *Groves class of mechanisms is DSIC.*

Proof: Suppose for agent i, the true type is θ_i and the reported type is $\hat{\theta}_i$. Also, assume that

$$f^{AE}(\theta_i, \theta_{-i}) = a, \text{ and } f^{AE}(\hat{\theta}_i, \theta_{-i}) = b.$$

Consider the utility of agent i when he reports θ_i

$$\begin{aligned}
& v_i(f^{AE}(\theta), \theta_i) - {p_i}^G(\theta) \\
&= v_i(f^{AE}(\theta), \theta_i) - h_i(\theta_{-i}) + \sum_{j \neq i} v_j(f^{AE}(\theta), \theta_j) \\
&= \sum_{j \in N} v_j(f^{AE}(\theta), \theta_j) - h_i(\theta_{-i}) \\
&\geqslant \sum_{j \in N} v_j(b, \theta_j) - h_i(\theta_{-i}) \\
&= v_i(f^{AE}(\hat{\theta}_i, \theta_{-i}), \theta_i) - h_i(\theta_{-i}) + \sum_{j \neq i} v_j(f^{AE}(\hat{\theta}_i, \theta_{-i}), \theta_j) \\
&= v_i(f^{AE}(\hat{\theta}_i, \theta_{-i}), \theta_i) - {p_i}^G(\hat{\theta}_i, \theta_{-i}).
\end{aligned}$$

Where the inequality follows from the the definition of f^{AE}. Hence we have proved the theorem. ■

27.4 The Vickrey-Clarke-Groves (VCG) Mechanism

An interesting mechanism in the Groves class is the VCG mechanism, named after Vickrey, Clarke, and Groves. This is also called the *pivotal* mechanism. We will discuss later about its pivotality. It is characterized by a specific $h_i(\theta_{-i})$ given as follows.

$$h_i(\theta_{-i}) = \max_{b \in A} \sum_{j \neq i} v_j(b, \theta_j),$$

and hence

$$p_i(\theta) = \max_{b \in A} \left[\sum_{j \neq i} v_j(b, \theta_j) \right] - \sum_{j \neq i} v_j(f^{AE}(\theta), \theta_j).$$

Note that, if the set of allocations A remains unchanged after removing agent i, e.g., in the public good allocation problem, then the payment above is always $\geqslant 0$. This shows that VCG mechanism for public goods gives *no subsidy* to any agent. As a consequence, it is obvious that in such a setting, the VCG mechanism is also *weakly budget balanced* (no-deficit), since $\sum_{i \in N} p_i(\theta) \geqslant 0$.