CS698W: Game Theory and Collective Choice **Jul-Nov 2017**

Lecture 26: October 13, 2017

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Disclaimer: *These notes aggregate content from several texts and have not been subjected to the usual scrutiny deserved by formal publications. If you find errors, please bring to the notice of the Instructor at swaprava@cse.iitk.ac.in.*

26.1 Recap

In the last lecture we discussed the uniform rule SCF and proved that the uniform rule SCF is PE, ANON, and SP and mentioned that the converse is also true, i.e., a PE, ANON, and SP SCF must necessarily be an uniform rule. We also looked at another restricted domain of quasi-linear preferences. In this domain, an SCF $F : \Theta \mapsto X$, where X is the set of the tuples $(a, \boldsymbol{\pi})$ with a being the allocation and $\boldsymbol{\pi} := (\pi_1, \ldots, \pi_n)$ being the vector of payments, can be decomposed as

$$F \equiv (f, \boldsymbol{p}).$$

Here the function $f : \Theta \mapsto A$ is called the *allocation function* and $\boldsymbol{p} := (p_i, i \in N \mid p_i : \Theta \mapsto \mathbb{R})$ the *payment function*.

We provide some examples of these functions to illustrate the domain.

26.2 Example of allocation functions

- Constant rule: allocation function is constant for all θ.

$$f^c(\theta) = a, \quad \text{for some } a \in A, \forall\ \theta \in \Theta.$$

- Dictatorial rule: the allocation that maximizes the valuation of a pre-determined player, whom we call the dictator.

$$f^D(\theta) \in \operatorname*{argmax}_{a \in A} v_d(a, \theta_d), \text{ for some } d \in N.$$

- Allocatively efficient (AE) rule / Utilitarian rule: allocation that maximizes the *social welfare*

$$f(\theta) \in \operatorname*{argmax}_{a \in A} \sum_{i \in N} v_i(a, \theta_i).$$

- Weighted efficient rule: a slight variant of the AE rule

$$f(\theta) \in \operatorname*{argmax}_{a \in A} \sum_{i \in N} w_i v_i(a, \theta_i), w_i \geqslant 0, \forall i \in N, \text{ (not all } w_i\text{'s are zero).}$$

- Max-Min (also called egalitarian or Rawlsian) allocation rule: maximizes the minimum valuation of the agents

$$f^R(\theta) \ \in \ \operatorname*{argmax}_{a \in A} \min_{i \in N} v_i(a, \theta_i).$$

- Affine maximizer rule: further generalization of the AE rule

$$f_{\mathrm{AM}}(\theta) \;\in\; \underset{a \;\in\; A}{\operatorname{argmax}} \left(\sum_{i \;\in N} \lambda_i v_i(a, \theta_i) + \kappa(a) \right), , \lambda_i \geqslant 0, \forall i \in N, \text{ (not all } \lambda_i\text{'s are zero).}$$

26.3 Examples of payment rules

- Weak budget balanced (also called no-deficit or feasible) payment rule:

$$\sum_{i \;\in\; N} p_i(\theta) \geqslant 0, \forall \theta \in \Theta.$$

  This payment rule makes sure that the mechanism designer does not need to add money to the system to run the mechanism. There could be a surplus, but never a shortage of total payment.

- No subsidy:

$$p_i(\theta) \geqslant 0, \forall i \in N, \forall \theta \in \Theta.$$

  This payment rule makes sure that every agent is asked to pay in the mechanism.

- Budget balanced:

$$\sum_{i \in N} p_i(\theta) = 0.$$

  This ensures that there is neither surplus nor deficit in the payments.

However, in the discussion that follows, we will be prioritizing the allocation rule, and will put less restriction on the payment rule. In particular, we will be interested in finding an allocation that is "truthful" and may put no restriction on payments.

26.4 Incentive compatibility

To denote *truthfulness* of a mechanism, we need to distinguish the true type with the reported type. Denote the reported type with $\hat{\theta} = (\hat{\theta}_1, \hat{\theta}_2, \ldots, \hat{\theta}_n)$. Recall that a direct mechanism is denoted by $((f, \boldsymbol{p}), \Theta)$, and the allocation and payment function are evaluated over the *reported* types. The utility of agent i is given by

$$v_i(f(\hat{\theta}), \theta_i) - p_i(\theta).$$

Definition 26.1 (Dominant Strategy Incentive Compatibility (DSIC)) *A direct mechanism* $(f, \boldsymbol{p})$ *is* dominant strategy incentive compatible *(DSIC) if,*

$$v_i(f(\theta_i, \theta_{-i}), \theta_i) - p(\theta_i, \theta_{-i}) \geqslant v_i(f(\hat{\theta}_i, \theta_{-i}), \theta_i) - p(\hat{\theta}_i, \theta_{-i})$$
$$\forall \theta_i, \hat{\theta}_i \in \Theta_i, \forall \theta_{-i} \in \Theta_{-i}, \forall i \in N.$$

Note: the definition of DSIC implies that reporting types truthfully is a WDSE. If the above mentioned conditions hold then we say that $\boldsymbol{p}$ implements f in dominant strategies.

Let us illustrate the conditions given by the definition above for a case with two agents and two types.

Example 26.1 *Consider* $N = \{1,2\}, \Theta_1 = \Theta_2 = \{\theta^H, \theta^L\}$. *The allocation rule is* $f : \Theta_1 \times \Theta_2 \mapsto A$. *If* $\boldsymbol{p}$ *implements* f *in dominant strategies, then for player 1, the conditions are*

$$v_1(f(\theta_H,\theta_2),\theta_H) - p(\theta_H,\theta_2) \geqslant v_1(f(\theta_L,\theta_2),\theta_H) - p(\theta_L,\theta_2), \forall\theta_2 \in \Theta_2,$$
$$v_1(f(\theta_L,\theta_2),\theta_L) - p(\theta_L,\theta_2) \geqslant v_1(f(\theta_H,\theta_2),\theta_L) - p(\theta_H,\theta_2),\ \forall\theta_2 \in \Theta_2.$$

Similarly for player 2

$$v_2(f(\theta_H,\theta_1),\theta_H) - p(\theta_H,\theta_1) \geqslant v_2(f(\theta_L,\theta_1),\theta_H) - p(\theta_L,\theta_1),\ \forall\theta_1 \in \Theta_1,$$
$$v_2(f(\theta_H,\theta_1),\theta_H) - p(\theta_H,\theta_1) \geqslant v_2(f(\theta_L,\theta_1),\theta_H) - p(\theta_L,\theta_1),\ \forall\theta_1 \in \Theta_1.$$

26.5 Impact of DSIC on payments

1. Effect of an additive function to payment: Let $(f,\boldsymbol{p})$ is DSIC. We have

$$v_i(f(\theta_i,\theta_{-i}),\theta_i) - p(\theta_i,\theta_{-i}) \geqslant v_i(f(\hat{\theta}_i,\theta_{-i}),\theta_i) - p(\hat{\theta}_i,\theta_{-i})$$
$$\forall\theta_i,\hat{\theta}_i \in \Theta_i, \forall\theta_{-i} \in \Theta_{-i}, \forall i \in N.$$

Now consider

$$q(\theta_i,\theta_{-i}) = p(\theta_i,\theta_{-i}) + h_i(\theta_{-i}).$$

Question: Is $(f,\boldsymbol{q})$ DSIC?

Answer: Yes it is. Consider

$$\begin{aligned} &v_i(f(\theta_i,\theta_{-i}),\theta_i) - q(\theta_i,\theta_{-i}) \\ &= v_i(f(\theta_i,\theta_{-i}),\theta_i) - p(\theta_i,\theta_{-i}) - h_i(\theta_{-i}) \\ &\geqslant v_i(f(\hat{\theta}_i,\theta_{-i}),\theta_i) - p(\hat{\theta}_i,\theta_{-i}) - h_i(\theta_{-i}) \\ &= v_i(f(\hat{\theta}_i,\theta_{-i}),\theta_i) - q(\hat{\theta}_i,\theta_{-i}). \end{aligned}$$

The first inequality holds because $(f,\boldsymbol{p})$ is DSIC and the equalities hold from the definition of q_i's.

2. If the allocation is same for two different types of agent i: Consider types θ_i and $\hat{\theta}_i$ of agent i and assume that for θ_{-i}

$$f(\theta_i,\theta_{-i}) = f(\hat{\theta}_i,\theta_{-i}) = a \text{ (say)}$$

Suppose $(f,\boldsymbol{p})$ is DSIC. When agent i's true type is θ_i, we have

$$v_i(f(\theta_i,\theta_{-i}),\theta_i) - p(\theta_i,\theta_{-i}) \geqslant v_i(f(\hat{\theta}_i,\theta_{-i}),\theta_i) - p(\hat{\theta}_i,\theta_{-i})$$
$$p_i(\hat{\theta}_i,\theta_{-i}) \geqslant p_i(\theta_i,\theta_{-i}). \qquad \text{(since } f(\theta_i,\theta_{-i}) = f(\hat{\theta}_i,\theta_{-i}).\text{)}$$

When agent i's true type is $\hat{\theta}_i$, similarly we have

$$v_i(f(\hat{\theta}_i,\theta_{-i}),\hat{\theta}_i) - p(\hat{\theta}_i,\theta_{-i}) \geqslant v_i(f(\theta_i,\theta_{-i}),\hat{\theta}_i) - p(\theta_i,\theta_{-i})$$
$$p_i(\theta_i,\theta_{-i}) \geqslant p_i(\hat{\theta}_i,\theta_{-i}).$$

Combining the above two observations we get

$$p_i(\theta_i,\theta_{-i}) = p_i(\hat{\theta}_i,\theta_{-i}).$$

Hence, if the allocation does not change by an unilateral deviation of type of an agent, the payment also does not change.