CS698W: Game Theory and Collective Choice	**Jul-Nov 2017**

Lecture 23: October 6, 2017

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Disclaimer: *These notes aggregate content from several texts and have not been subjected to the usual scrutiny deserved by formal publications. If you find errors, please bring to the notice of the Instructor at swaprava@cse.iitk.ac.in.*

23.1 Recap

In the previous lecture we showed some important properties of social choice function in the restricted domain of single-peaked preferences. The claims we proved are as follows.

1. Let p_{min} and p_{max} are the leftmost and rightmost peaks according to order relation $<$. Then SCF f is PE if and only if $f(P) \in [p_{min}, p_{max}]$.
2. f is SP $\implies$ f is MONO.
3. Let $f : \mathcal{S}^n \to A$ is SP. Then f is ONTO $\iff$ f is UN $\iff$ f is PE.

We also defined anonymous (ANON) SCF f which is independent of the permutation of the agents for every preference profile P, that is, $f(P) = f(P^\sigma)$ where P^σ represents σ-permuted preferences of P. We observed that a dictatorial SCF cannot be ANON.

23.2 Characterization of strategyproof SCFs in single-peaked domain

We started proving a characterization result for the median voting rule SCF given as follows.

Theorem 23.1 (Moulin 1980) *A SP SCF f is ONTO and ANON if and only if it is a median voting rule.*

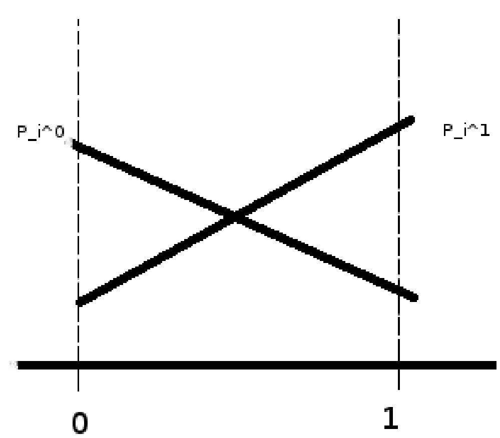

Figure 23.1: Special single peaked preferences over $[0, 1]$ – P_i^0 and P_i^1.

Proof: (Continued from the last lecture)
Consider an arbitrary profile

$$P = (P_1, P_2, \ldots, P_n)$$
$$\text{Let } p_i := P_i(1) \text{ denote the peak of agent } i$$

We claim that $f(P) = med(p_1, p_2, \ldots, p_n, y_1, \ldots, y_{n-1})$.
We can assume WLOG that $p_1 \leqslant p_2 \leqslant \ldots \leqslant p_n$ due to ANON. Say $a = med(p_1, p_2, \ldots, p_n, y_1, \ldots, y_{n-1})$.

Case 1: a is a phantom peak
Say $a = y_j$ for some $j \in 1, 2, \ldots, n-1$. This is a median of $(2n-1)$ points. There are $(j-1)$ phantom peaks to the left of the median (due to the fact that $y_j \leqslant y_{j+1}$) and $(n-1-j)$ to the right. So, there are $(n-j)$ agent peaks on the left. Hence the following holds,

$$p_1 \leqslant \ldots \leqslant p_{n-j} \leqslant y_j = a \leqslant p_{n-j+1} \leqslant \ldots \leqslant p_n.$$

Now consider two profiles, $(P_1^0, P_2^0, \ldots, P_{n-j}^0, P_{n-j+1}^1, \ldots, P_n^1)$ and $(P_1, P_2^0, \ldots, P_n^1)$. By definition

$$f(P_1^0, P_2^0, \ldots, P_{n-j}^0, P_{n-j+1}^1, \ldots, P_n^1) = y_j.$$

Assume that
$$f(P_1, P_2^0, \ldots, P_n^1) = b$$

Now we see that

$$f \text{ is SP} \implies y_j P_1^0 b \implies y_j \leqslant b.$$

But also

$$f \text{ is SP} \implies b P_1 y_j \text{ and it is known that } p_1 \leqslant y_j$$
$$\implies b \leqslant y_j$$

Combining the above two implications we get, $b = y_j$. Repeating the argument for the first $(n-j)$ agents, we get
$$f(P_1, P_2, \ldots, P_{n-j}, P_{n-j+1}^1, \ldots, P_n^1) = y_j.$$
Now consider $f(P_1, \ldots, P_{n-j}, P_{n-j+1}^1, \ldots, P_{n-1}^1, P_n) = b$ (say). Using the SP property of f, we get

$$y_j P_n^1 b \implies b \leqslant y_j$$
$$b P_n y_j \text{ and } y_j \leqslant p_n \implies y_j \leqslant b$$

Combining the above two implications, $b = y_j$. Repeating the arguments, we finally get,

$$f(P_1, P_2, \ldots, P_{n-j}, P_{n-j+1}, \ldots, P_n) = y_j = a.$$

which is the median.

Case 2: a is an agent peak
We prove this for 2 agents. The general case repeats the argument.

Claim 23.2 *Let $N = \{1, 2\}$, and P, P' be such that, $P_i(1) = P_i'(1)$ $\forall i \in N$, then*

$$f(P) = f(P').$$

Proof: Let $a = P_1(1) = P'_1(1)$ and $b = P_2(1) = P'_2(1)$. Also let $f(P) = x$ and $f(P'_1, P_2) = y$. Since f is SP, we have xP_1y and yP'_1x. Since peaks in the two profiles are the same, if x and y fall on the same side of the peak $P_1(1)$ (equivalently $P'_1(1)$) they must be the same. The only other possibility is that x and y fall on the different sides of the peak. We show that this is not possible.
WLOG assume that, $x < a < y$ and $a < b$. We know f is SP+ONTO $\iff$ f is SP+PE and PE requires that $f(P) \in [a, b]$. But $f(P) = x < a$, which is a contradiction.
Repeat this argument for the transition of preference profiles $(P'_1, P_2) \to (P'_1, P'_2)$. ■
Now consider the profile $P = (P_1, P_2)$ such that $P_1(1) = a$ and $P_2(1) = b$ and y_1 be the phantom peak. By assumption, $med(a, b, y_1)$ is an agent peak. WLOG let the median be a. Assume for contradiction, $f(P) = c \neq a$.
By PE, c must lie within a and b. We consider the two cases, $b < a < y_1$ and $y_1 < a < b$.
Case A: $b < a < y_1$
By PE, $c < a$. Construct P'_1 such that $P'_1(1) = a = P_1(1)$ and $y_1P'_1c$ (possible since y_1 and c are on different sides of the peak $P'_1(1)$). Since $f(P) = c$, $f(P'_1, P_2) = c$ by the previous claim.
Now consider the profile (P^1_1, P_2). We have

$$P_2(1) = b < y_1 < P^1_1(1).$$

So the median of (b, P^1_1, y_1) is y_1, which is a phantom peak, and hence by our result in Case 1,

$$f(P^1_1, P_2) = y_1.$$

By construction of P'_1,

$$\begin{aligned} y_1P'_1c &\implies f(P^1_1, P_2)\ P'_1\ f(P'_1, P_2) \\ &\implies f \text{ is not SP.} \end{aligned}$$

This is a contradiction. Hence our assumption $f(P) \neq a$ is wrong in this case.
Case B: $y_1 < a < b$
By PE, $a < c$. Construct P'_1 such that $P'_1(1) = a = P_1(1)$ and $y_1P'_1c$ (possible since y_1 and c are on different sides of the peak $P'_1(1)$). By the previous claim

$$f(P) = c \implies f(P'_1, P_2) = c.$$

Now consider the profile (P^0_1, P_2). We have

$$P^0_1(1) < y_1 < b = P_2(1) \implies f(P^0_1, P_2) = y_1.$$

But by construction of P'_1,

$$\begin{aligned} y_1P'_1c &\implies f(P^0_1, P_2)\ P'_1\ f(P'_1, P_2) \\ &\implies f \text{ is not SP.} \end{aligned}$$

This is a contradiction. Hence our assumption $f(P) \neq a$ is wrong in this case too.
Hence we have proved Case 2 of this theorem for 2 agents. ■

23.3 Conclusion

In this lecture, we have proved the non dictatorial nature of median voter SCF by introducing phantom voters. The phantom voters/peaks are introduced so that the extreme preference conditions can be handled with a "fair" decision. For example, if half the agents are at the extreme left and other half is at the extreme right, a fair distribution of phantom peaks may lead to picking the median somewhere at the center rather than at some extreme point. Note that, median voter SCF is actually a class of voting rules.