CS698W: Game Theory and Collective Choice **Jul-Nov 2017**

Lecture 22: October 4, 2017

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Disclaimer: *These notes aggregate content from several texts and have not been subjected to the usual scrutiny deserved by formal publications. If you find errors, please bring to the notice of the Instructor at swaprava@cse.iitk.ac.in.*

22.1 Properties of SCF (Recap)

In previous lecture we discussed about the restricted domains (single peaked preferences in particular). Now, we discuss properties in the single-peaked domain, some of which are similar to the properties of unrestricted domain. Denote the set of single peaked preferences over the alternatives in A with $\mathcal{S}$.

Definition 22.1 (Pareto Efficiency (PE)) *$\forall\ P \in \mathcal{S}^n$, $\forall\ b \in A$ if $\exists\ a \in A$ such that $aP_ib\ \forall i \in N$, then $f(P) \neq b$.*

Definition 22.2 (Unanimity (UN)) *$\forall\ P$ with $P_1(1) = P_2(1) = \ldots = P_n(1) = a$, then $f(P) = a$.*

Definition 22.3 (Onto-ness (ONTO)) *$\forall\ a \in A$, $\exists\ P \in \mathcal{S}^n$ such that $f(P) = a$.*

As before, PE $\subseteq$ UN $\subseteq$ ONTO.

Claim 22.4 *Let $p_{\min}$ and $p_{\max}$ are the leftmost and rightmost peaks of P according to $<$, then f is PE iff $f(P) \in [p_{\min}, p_{\max}]$.*

Proof: ($\Rightarrow$) Suppose f is PE but $f(P) \notin [p_{\min}, p_{\max}]$, then $f(P)$ is either at the left side of $p_{\min}$ or at the right side of $p_{\max}$. Consider the case $f(P) < p_{\min}$, but then every agent prefers $p_{\min}$ over $f(P)$, a contradiction to PE. Similar argument can be given for $p_{\max} < f(P)$.

($\Leftarrow$) if $f(P) \in [p_{\min}, p_{\max}]$, every other alternative $b \neq f(P)$ that belongs to $[p_{\min}, p_{\max}]$ will either be close to $p_{\min}$ (and further from $p_{\max}$) or vice-versa. Then there exists at least one agent that prefers $f(P)$ more than b. Hence, the 'if' condition in the definition of PE is never triggered for every such b. Hence, PE is vacuously satisfied. ■

Definition 22.5 (Monotonicity) *If for two profiles P and P' with $f(P) = a$ and $D(a, P_i) \subseteq D(a, P'_i)$, $\forall\ i \in N$, then $f(P') = a$.*

- where $D(a, P_i) = \{b \in A : aP_ib\}$ is the set of alternatives dominated by a under the preference P_i.

22.2 Results on restricted domain of preferences

We will see some results similar to the unrestricted preferences but the proofs will differ as we do not have the flexibility of constructing arbitrary preference profiles. In this lecture, we consider only single peaked preferences.

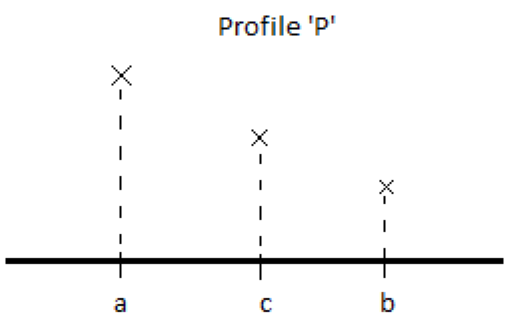
Theorem 22.6 *f is strategyproof (SP) $\Rightarrow$ f is monotone (MONO).*

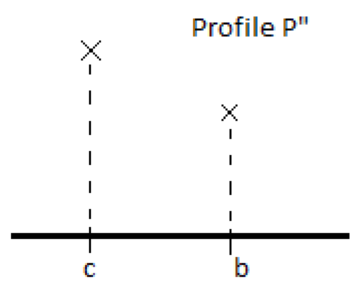
This has exactly the same proof as we did for unrestricted preferences. However the construction of our previous result is not always feasible for the converse of this theorem.

Exercise: Find a counterexample of the converse OR prove the converse.

Theorem 22.7 *Let an SCF $f : \mathcal{S}^n \mapsto A$ be SP, then f is ONTO $\iff$ f is UN $\iff$ f is PE.*

Proof: It is enough to show that if f is SP, f is ONTO $\Rightarrow$ f is PE. We prove this by contradiction. Suppose $\exists$ a, b such that aP_ib, $\forall$ $i \in N$ but $f(P) = b$. Since P_i is single peaked, $\exists$ another alternative $c \in A$ which





is adjacent to b (there is no alternative between c and b) such that cP_ib, $\forall$ $i \in N$. Note that, c could be a itself. This is illustrated in the figure above.

ONTO implies that $\exists$ P' such that $f(P') = c$. Construct P'' such that $P''_i(1) = c$ and $P''_i(2) = b, \forall i \in N$. Note that this profile is always possible to construct in a single-peaked domain. [can you think of a similar construction for proving the converse of theorem 22.6?]

Consider the transition of the profile from $P' \longrightarrow P''$, by monotonicity $f(P'') = c$. Now consider the transition of the profile from $P \longrightarrow P''$, by monotonicity $f(P'') = b$. This is a contradiction since $c \neq b$. ■

22.3 Anonymity

Define permutation over the agents as $\sigma : N \mapsto N$. We apply a permutation σ to a profile P to construct another profile such that the preference ordering of i in P goes to the permuted agent $\sigma(i)$ in the new profile. We denote the new profile by P^σ.

Example: $N = \{1, 2, 3\}$, $\sigma : \sigma(1) = 2, \sigma(2) = 3, \sigma(3) = 1$.

P_1	P_2	P_3	P_1^σ	P_2^σ	P_3^σ
a	b	b	b	a	b
b	a	c	c	b	a
c	c	a	a	c	c

Anonymity requires that the social outcome should not depend on agent identity.

Definition 22.8 *An SCF $f : \mathcal{S}^n \mapsto A$ is anonymous (ANON) if for every profile P and for every permutation of the agents σ, $f(P^\sigma) = f(P)$.*

Note: Dictatorship is not anonymous, but median voter rule is anonymous.

Theorem 22.9 (Moulin 1980)) *A SP SCF f is ONTO and ANON iff it is a median voter SCF.*

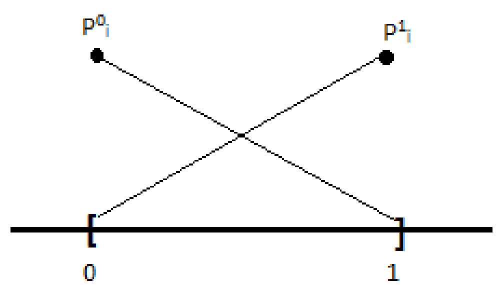
Proof: ($\Leftarrow$) Median voter SCF is SP (see theorem 21.9 of previous lecture), and

- it is ONTO, since we can put all voters peaks at the same alternative/location and set all phantom peaks at zero, then that location is the outcome.
- it is ANON, since we can permute the agents with the peaks unchanged and the outcome will not change.

($\Rightarrow$) Given $f : \mathcal{S}^n \mapsto A$ is SP, ONTO and ANON. Define

- P_i^0 : agent i's preference where the peak is at the leftmost point w.r.t. $<$.
- P_i^1 : agent i's preference where the peak is at the rightmost point w.r.t. $<$.

The preferences are illustrated in the figure below.



Let y_j's be the phantom peaks, $j = 1, 2, \ldots, n-1$. Pick y_j's as follows.

$$y_j = f(P_1^0, P_2^0, \ldots, P_{n-j}^0, P_{n-j+1}^1, \ldots, P_n^1).$$

It does not matter which agents have which peaks because of anonymity.

Claim 22.10 $y_j \leqslant y_{j+1}$, $j = 1, 2, \ldots, n-2$.

Proof: Consider y_j as defined before. Therefore,

$$y_{j+1} = f(P_1^0, P_2^0, \ldots, P_{n-j-1}^0, P_{n-j}^1, P_{n-j+1}^1, \ldots, P_n^1).$$

Strategyproofness implies $y_j P^0_{n-j} y_{j+1}$ but P^0_{n-j} is single peaked with the peak at the leftmost position. Therefore $y_j \leqslant y_{j+1}$. ■

We will complete this proof in the next lecture. ■