CS698W: Game Theory and Collective Choice **Jul-Nov 2017**

Lecture 21: October 3, 2017

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21.1 Recap and need for domain restriction

In the previous lectures, we saw a characterization theorem due to Gibbard and Satterthwaite that is widely regarded as a negative result.

Theorem 21.1 (Gibbard (1973), Satterthwaite (1975)) *Let the set of alternatives A be such that $|A| \geqslant 3$. If the social choice function $f : \mathcal{P}^n \mapsto A$ is onto and strategyproof then f is dictatorial.*

Note that the GS theorem needs unrestricted preferences. One of the reasons of such a restrictive result is that since the domain of the SCF is large, it leaves more opportunities for a potential manipulator. Revisiting the major assumptions used by theorem 21.1 can help us better understand this:

- *(Unrestricted preferences)* For $|A| = m \geq 3$, $|\mathcal{P}| = m!$, i.e., all possible linear orderings are available to be chosen by the agents.
- *(Strategyproofness)* We require the SCF to satisfy

$$\forall i \in N, \forall P_i', P_i \in \mathcal{P}, \forall P_{-i} \in \mathcal{P}^{n-1},$$
$$f(P_i, P_{-i}) P_i f(P_i', P_{-i}) \text{ or } f(P_i, P_{-i}) = f(P_i', P_{-i}). \tag{21.1}$$

If we now reduce the domain of the SCF from the set of linear orderings (over A), $\mathcal{P}^n$, to some subset of $\mathcal{P}$, the SCFs that are truthful with domain $\mathcal{P}^n$ will continue to be truthful on the subset. However, we can hope to find more SCFs that are truthful (and potentially non-dictatorial) on the new restricted domain. In this and the following lectures, we will look at some of these restricted domains.

21.2 Restricted domains

In this course, we will be studying the following three domain restrictions:

1. Single-peaked preferences
2. Divisible object allocation
3. Quasilinear preferences (sometimes, also referred to as 'mechanisms with money')

It is worth stating that each of these subdomains has interesting *non-dictatorial* but *strategyproof* SCFs defined on it. We will be first dealing with single-peaked preferences.

21.3 Single-peaked preferences

In this restricted domain, we set a single common order over the alternatives. Once the common order is chosen, we allow only those preferences from $\mathcal{P}$ which have a single peak with respect to that common order. We will soon be defining such preferences formally for a particular common order.
Note: The common order is a fixed order relation over the alternatives and is common to all agents. It is fixed before the agents pick their preferences.

21.3.1 Motivating examples

1. Facility location: placing a public facility like a school, hospital, or post-office in a city.
2. Political ideology: individual political opinions that can be either left, center, or right.
3. Temperature sensing: an individual is most comfortable at a specific temperature and anything hotter or colder is less preferred.

Each of these examples have a natural ordering over the alternatives. We may place the alternatives on the real line $\mathbb{R}$ according to the common ordering $<$. The ordering need not always be an order relation on the real numbers, but can be any order relation that is *transitive* and *anti-symmetric*. For simplicity, we will consider only alternatives having an order on the real line.

21.3.1.1 An illustration

Let $A := \{a, b, c\}$ be the set of facilities ordered on a real line s.t. $a < b < c$ (referring to the locations of a, b, c). Note, here $|\mathcal{P}| = 6$ and the following table of valid and invalid preferences illustrates that for the set of single-peaked preferences $\mathcal{S}$, $|\mathcal{S}| = 4$. Also, clearly $\mathcal{S} \subset \mathcal{P}$

Table 21.1: Preferences

(a) Valid

a	b	b	c
b	a	c	b
c	c	a	a

(b) Invalid

a	c
c	a
b	b

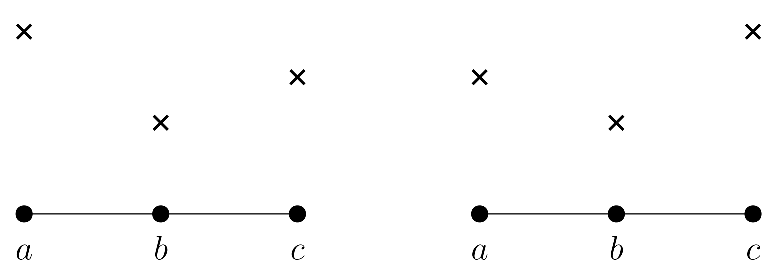

Figure 21.1: Invalid preference ordering in single-peaked domain.

We now formally define single-peaked preferences.

Definition 21.2 (Single-peaked preference ordering) *A preference ordering P_i (strict order over A) for agent i is single-peaked w.r.t. the common order $<$ over the alternatives if there exists an alternative in the agent's preference denoted by $P_i(1)$ such that*

- *$\forall b, c \in A$ with $b < c \leqslant P_i(1)$, we have cP_ib, and*
- *$\forall b, c \in A$ with $P_i(1) \leqslant b < c$, we have bP_ic.*

The alternative $P_i(1)$ is called the peak of the preference of agent i.

Let $\mathcal{S}$ be the set of single-peaked preferences. As illustrated in the earlier example of this section, $\mathcal{S} \subset \mathcal{P}$. The SCF on this restricted domain is $f : \mathcal{S}^n \mapsto A$.

Definition 21.3 (Manipulability of an SCF) *An SCF f is said to be manipulable if $\exists\ i \in N, P_i', P_i \in \mathcal{S}, P_{-i} \in \mathcal{S}^{n-1}$ such that*

$$f(P_i', P_{-i})\ P_i\ f(P_i, P_{-i}).$$

Definition 21.4 (Strategyproof) *An SCF f is said to be strategyproof if it is not manipulable. Thus, by definition, it implies*

$$\begin{aligned}&\forall i \in N, \forall P_i', P_i \in \mathcal{S}, \forall P_{-i} \in \mathcal{S}^{n-1},\\ &f(P_i, P_{-i}) P_i f(P_i', P_{-i}) \quad or \quad f(P_i, P_{-i}) = f(P_i', P_{-i}).\end{aligned} \tag{21.2}$$

Note that in contrast to Eq. 21.1, now the preferences $P_i', P_i \in \mathcal{S}$, $P_{i-} \in \mathcal{S}^{n-1}$ and $\mathcal{S} \subset \mathcal{P}$. This implies that we have less number of conditions to satisfy for strategyproofness.

The question is, how does an SCF defined on the restricted domain of single-peaked preferences circumvent the Gibberd-Sattherthwaite theorem? We illustrate this using an example social choice function.

21.3.2 Strategyproof SCFs

Definition 21.5 *Define SCF $f : \mathcal{S}^n \mapsto A$, where $\mathcal{S}$ is the set of single-peaked preferences w.r.t. the common order $<$ as, for $P \in \mathcal{S}^n$*

$$f(P) = \min_{i \in N}\{P_i(1)\}.$$

Where minimum is taken w.r.t. the order relation $<$. Hence the SCF picks the left-most peak among the peaks of the agents.

Claim 21.6 *The SCF f defined above is strategyproof and non-dictatorial.*

Proof: For proving strategyproofness, first consider the agent having the peak preference as the left-most alternative. Clearly, she has no reason to misreport her preference order since her peak is chosen.

Now, consider any other agent i – she has a peak to the right of the left-most peak. In other words, for $i \in N$ we have $f(P) < P_i(1)$. The only possible manipulation she could do to change the outcome is to report her peak to be further left of $f(P)$, therefore changing the preference profile from $(P_i, P_{-i}) \to (P_i', P_{-i})$ s.t.

$$\begin{aligned}&P_i'(1) \leqslant f(P) < P_i(1),\\ &f(P_i', P_{-i}) = \min_{i \in N}\{P_1(1), P_2(1), .., P_i'(1), , , , P_n(1)\} = P_i'(1).\end{aligned}$$

If $P_i'(1) = f(P)$, clearly $f(P_i, P_{-i}) = f(P_i', P_{-i})$. Else, we will have

$$P_i'(1) < f(P) < P_i(1).$$

Since P_i is a single-peaked preference, thus, we get $f(P_i, P_{-i})\ P_i\ f(P_i', P_{-i})$. Hence, by definition 21.4, we conclude that f is strategyproof.

Since the identity of the agent having the left-most peak is not fixed before reporting the preferences, f is non-dictatorial. ■

Using similar arguments, we can prove that an SCF that picks the k^{th} alternative from the left will also be *strategyproof* and *non-dictatorial*, $\forall k \in \{1, 2, .., |A|\}$. In particular, it holds true for the right-most alternative and median ($k = \lfloor n/2 \rfloor$).

Definition 21.7 (Median Voter SCF) *An SCF $f : \mathcal{S}^n \mapsto A$ is said to be a Median Voter SCF if $\exists\ B = (y_1, y_2, .., y_{n-1})$ s.t. $f(P) = median(B, peaks(P)), \forall P \in \mathcal{S}^n$. The points in B are called as "peaks of phantom voters" or "phantom peaks".*

Note: B is fixed for a given f and does not change with $P \in \mathcal{S}^n$.

21.3.2.1 Advantage of using phantom voters

All the examples stated above, i.e., left-most peak, right-most peak, median, or any k-th peak from the left etc. can be combined into a single definition through this *median voter SCF*.

Claim 21.8 *The SCFs picking the left-most most peak, the right-most peak are median voter SCFs.*

Proof: If $A = \{a_1, a_2, .., a_{|A|}\}$, let $a = \min_{w.r.t. <} A, b = \max_{w.r.t. <} A$
Define $y_1, y_2, .., y_{n-1}, z_1, z_2, .., z_{n-1} \in \mathcal{S}^n$ s.t.

$$y_i(1) = a, \forall i \in \{1, \ldots, n-1\},$$
$$z_i(1) = b, \forall i \in \{1, \ldots, n-1\}.$$

For the case of the left-most peak SCF, we can choose $B = (a, a, ..., a)$ which ensures that the median of points in B and peaks reported by the agents will always result in the minimum of the peaks w.r.t $<$ reported by the agents. For the case of right-most SCF, we can choose $B = (b, b, ..., b)$ and the proof follows as for the case of minimum. ■

Using similar arguments, we can prove that the other SCFs picking any k^{th} peak from left are also Median Vector SCFs.

Theorem 21.9 (Moulin(1980)) *Every median vector SCF is strategyproof.*

Proof: We need to consider only the peak preferences of all the agents. So let us denote the preferences denoted only by their peaks, i.e., $P = (P_1(1), .., P_i(1), .., P_n(1))$ and let $f(P) = a \in A$ is the median of these peaks and the phantom peaks. Consider an agent i

- If $P_i(1) = a$, then there is no reason for i to manipulate.

- If $P_i(1) < a$, then if the agent shifts her preference to further left of a, the median will not change. If she manipulates to report her peak to further right of a, *i.e.* $(P_i, P_{-i}) \mapsto (P'_i, P_{-i})$ s.t. $a < P'_i(1)$, this will imply that $P_i(1) < a < P'_i(1)$, and since P_i is a single-peaked preference, by definition 21.4, $a = f(P_i, P_{-i})\ P_i\ f(P'_i, P_{-i})$. Thus, i has no profitable manipulation.
- If $a < P_i(1)$, again by exactly symmetrical arguments, i has no profitable manipulation.

Hence, f is strategyproof. ■

Note: Mean does not satisy the property used in the proof above, since changing the peak $P_i(1)$ of agent i on either sides of a will result in a change in the mean.

21.4 Summary

In this lecture, we revisited the Gibbard-Satterthwaite theorem and due to the restrictiveness of the result, we emphasized the need for domain restriction. Among the major domain restrictions, we looked at the single peaked preferences. In particular, we looked at examples of SCFs that are *strategyproof* but *non-dictatorial* and defined the median voter SCF which characterizes a set of these examples.