CS698W: Game Theory and Collective Choice **Jul-Nov 2017**

Lecture 19: September 13, 2017

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19.1 Recap

In the last lecture we started motivating the social choice setting with some real examples like voting. As the setting has changed, we need to redefine the axioms because we observe that IIA does not make sense as there is no social ordering anymore. In this lecture we define some new axioms and we notice some similarity between these axioms and the axioms defined in the social welfare setting. Recall the definition of social choice function.

Definition 19.1 *A social choice function f is a mapping from the set of strict preference profiles ($\mathcal{P}^n$) to the set alternatives A, i.e.,*

$$f : \mathcal{P}^n \mapsto A.$$

The output of the social choice function is an alternative instead of an ordering.

19.2 Definitions

We define the following properties in the social choice setting.

Definition 19.2 (Pareto Domination) *In a preference profile P, an alternative a is Pareto dominated by b if $bP_ia, \ \forall \ i \in N$.*

We see that Pareto domination is defined for a particular preference profile. Our next property is a property of social choice function which states that if an alternative a is dominated, then the social outcome should not be equal to a. Formally:

Definition 19.3 (Pareto Efficiency) *An SCF f is* Pareto efficient *(PE) if $\forall P \in \mathcal{P}$, if a is Pareto dominated in P then $f(P) \neq a$.*

An SCF f is unanimous if the most preferred alternatives of all agents are same in a preference profile, then the social choice outcome in that profile must be equal to that alternative. Formally,

Definition 19.4 (Unanimity) *An SCF f is* unanimous *(UN) if $\forall P$ such that $P_1(1) = P_2(1) = \ldots = P_n(1) = a$, then $f(P) = a$.*

In the above definition the subscripts correspond to the agent and arguments in the parentheses correspond to the rank of the alternative in the preference of the agent.

From the definitions of Pareto efficiency and unanimity we can claim that PE implies UN. With a little abuse of notation, we denote the set of all PE SCFs as **PE** and UN SCFs as **UN**.

Claim 19.5 ***PE*** $\subset$ ***UN***.

Proof: Suppose $F \in \textbf{PE}$. Pick an arbitrary P such that $P_1(1) = P_2(1) = \ldots = P_n(1) = a$, i.e., the 'if' condition of UN holds. Clearly, this implies that $aP_ib, \ \forall b \in A \setminus \{a\}, \ i \in N$. Since every $b \in A \setminus \{a\}$ is Pareto dominated by a, $F(P) \neq b$ by PE. Hence we conclude that $F(P) = a$. Hence F is UN. ∎

Note that the containment is strict.

Strict Example: When the top alternatives for all the agents in a preference profile are not the same, UN does not enforce any outcome. In such a case, one can choose an alternative that is strictly dominated to make the SCF not PE.

The next property is onto-ness, which states that for every alternative there exists a preference profile whose social outcome is that alternative.

Definition 19.6 (Onto) *An SCF F is* onto *(ONTO) if $\forall a \in A$, $\exists P \in \mathcal{P}^n$ such that $f(P) = a$.*

Similar to our earlier notation, we denote the set of all onto SCFs as **ONTO**.

Claim 19.7 ***UN*** $\subset$ ***ONTO***.

Proof: Since $\mathcal{P}$ contains all possible preferences over the alternatives, for every $a \in A$, there exists preference orders where a is on top. Consider a preference profile P where every agent has this same preference order having a on top – UN implies that $F(P) = a$. Hence F is onto. ∎

Strict example: Consider an SCF defined as follows. Consider a specific sequence of the alternatives, say WLOG $(a_1, a_2, \ldots, a_m)$. For every profile with all agents having the same top alternative, say a_k the SCF picks a_{k+1}, for $k = 1, \ldots, m-1$, and a_1 for $k = m$. This is clearly ONTO, since all alternatives are chosen by the SCF for some profile, but not UN.

19.2.1 Truthfulness

In the setting of social choice, there is no notion of IIA. Rather we will consider the property of truthfulness. It is similar to the definition of dominant strategy incentive compatibility where it is defined in terms of utility representation. However, since here we talk in terms of ordinal preference profiles, the definition will be adapted accordingly.

Definition 19.8 *An SCF is* manipulable *if $\exists \ i \in N, P_i, P_i' \in \mathcal{P}, P_{-i} \in \mathcal{P}^{n-1}$ such that $f(P_i', P_{-i}) P_i f(P_i, P_{-i})$.*

Therefore, an SCF is manipulable when an agent at some profile P is strictly better off by reporting a preference P_i' rather than her true preference P_i.

The SCF f is *non-manipulable* or *strategyproof* (SP) if it is not manipulable by any agent at any profile.

19.2.2 Characterization of strategyproofness

There exists a notion similar to IIA in the context of social choice functions that connects together different preference profiles, for which we need to define dominated sets.

Definition 19.9 (Dominated Set) *The* dominated set *of an alternative a at a preference ordering P_i is defined as $D(a, P_i) = \{b \in A : aP_ib\}$*

Hence, the dominated set is the collection of all such alternatives that are less preferred than a for a given preference ordering P_i.

Example: consider the following preference profile P_i

$$P_i = \begin{matrix} a \\ b \\ c \\ d \end{matrix} \qquad \text{then } D(b, P_i) = \{c, d\}. \tag{19.1}$$

Now we define a structural property that characterizes strategyproofness.

Definition 19.10 (Monotonicity) *An SCF f is* monotone *(MONO) if the profiles P and P' be such that $f(P) = a$ and $D(a, P_i) \subseteq D(a, P'_i)$, $\forall i \in N$, then $f(P') = a$.*

Illustration: consider two preference profiles P and P' as shown below and suppose that we have $f(P) = a$

$$\begin{array}{ccc} & P & \\ \hline a & & \\ . & & \\ . & a & a \\ . & . & . \\ . & . & . \end{array} \qquad \begin{array}{ccc} & P' & \\ \hline a & & \\ . & a & a \\ . & . & . \\ . & . & . \\ . & . & . \end{array} \tag{19.2}$$

That is, the relative position of a is getting weakly better from P to P', which implies that the alternatives that a was dominating in P are expanding in P'. Monotonicity requires that the social choice outcome should remain the same, i.e., $f(P') = a$.

Theorem 19.11 *An SCF f is SP if and only if f is MONO.*

Note: The technique used in the proof is used for proving other results in social choice.

Proof:

Part 1: SP $\Rightarrow$ MONO: Let f be a strategyproof SCF. Consider two profiles P and P' such that $f(P) = a$ and $D(a, P_i) \subseteq D(a, P'_i) \forall i \in N$. To show that f is monotone, we break the transition from P to P' into n stages such that in each stage the preference of exactly one agent changes as follows.

$$\begin{array}{ccccccccc} (P_1, P_2, \cdots, P_n) & \to & (P'_1, P_2, \cdots, P_n) & \to & (P'_1, P'_2, P_3 \cdots, P_n) & \to & (P_1, P_2, \cdots, P'_{k-1}, P'_k, \cdots, P_n) \\ P = P^{(0)} & & P^{(1)} & & P^{(2)} & \cdots & P^{(k)} \\ & & & & & & \downarrow \\ & & & & & & (P'_1, P'_2, \cdots, P'_n) \\ & & & & & \cdots & P^{(n)} = P' \end{array} \tag{19.3}$$

We claim that the social outcome should remain the same in these transitions.

Claim 19.12 $f(P^{(k)}) = a, \ \forall k = 1, 2, \cdots n.$

Since $P^{(n)} = P'$, this claim proves that f is monotone.

Proof: We prove the claim by contradiction. Suppose $\exists\ k$ such that

$$\begin{aligned} f\left(P^{(k-1)}\right) &= a, \quad \text{where } (P'_1, P'_2, \cdots, P'_{k-1}, P_k, \cdots, P_n) = P^{(k-1)} \text{ and,} \\ f\left(P^{(k)}\right) &= b \neq a, \quad \text{where } (P'_1, P'_2, \cdots, P'_k, P_{k+1}, \cdots, P_n) = P^{(k)}. \end{aligned} \tag{19.4}$$

Given the if part of monotone property, i.e., by moving from P to P' the relative position of a is weakly increasing there are three different possible cases.

- Case 1: aP_kb and aP'_kb: then at profile $P^{(k)}$ player k is better off by reporting P_k – thereby securing the outcome to be a.
- Case 2: bP_ka and bP'_ka: then at profile $P^{(k-1)}$ player k is better off by reporting P'_k – thereby securing the outcome to be b.
- Case 3: bP_ka and aP'_kb: then player k will manipulate in both profiles – at profile $P^{(k-1)}$, k is better off by reporting P'_k – thereby securing the outcome to be b, and at profile $P^{(k)}$, k is better off by reporting P_k – thereby securing the outcome to be a.

All the three cases are contradictions to f being strategyproof. Hence we have proved the claim. ■

Part 2: SP $\Leftarrow$ MONO: We prove this as !SP $\Rightarrow$!MONO. Say for contradiction, $\exists\ f$ which is not strategyproof but is monotone. Non-strategyproofness implies that f is manipulable, i.e., $\exists\ i, P_i, P'_i, P_{-i}$ such that $f(P'_i, P_{-i}) P_i f(P_i, P_{-i})$. Denote the social outcomes as $f(P'_i, P_{-i}) = b$ and $f(P_i, P_{-i}) = a$.

Construct another preference profile P'' such that $P'' = (P''_i, P_{-i})$ and also P''_i has the alternatives b and a in the top two positions as shown below.

$$\begin{array}{ccccl} P'' & = & (P''_i & , P_{-i}) & \\ & & b & . & \\ & & a & . & P''_i(1) = b, \text{ and } P''_i(2) = a. \\ & & . & . & \\ & & . & . & \end{array} \tag{19.5}$$

Now we look at the following two transitions and apply the definition of monotonicity.

- Transition $P \rightarrow P''$: note that in P, for player i, bP_ia and in P''_i, since b and a takes the top two positions respectively, it is clear that a's position has weakly improved. For the other agents, the preferences did not change. Hence

  $$D(a, P_j) \subseteq D(a, P''_j), \ \forall j \in N.$$

  Since f is monotone, we have $f(P'') = a$.
- Transition $P' \rightarrow P''$: Since b is at the top position in P''_i, and for the other agents, the preferences did not change, we have

  $$D(b, P'_j) \subseteq D(b, P''_j) \forall j \in N.$$

  Monotonicity of f gives $f(P'') = b \neq a$ which is a contradiction.

From Parts 1 and 2, we have the theorem. ■

We have seen that **PE** $\subset$ **UN** $\subset$ **ONTO**. However, the next result shows that they are equivalent under strategyproofness.

Lemma 19.13 *If f is monotone and onto then it is Pareto efficient.*

Proof: Assume for contradiction that f is monotone and onto but not Pareto efficient. Hence

$$\exists\ a, b, P \text{ such that } bP_i a, \forall i \in N \text{ but } f(P) = a.$$

Since f is onto

$$\exists\ P' \text{ such that } f(P') = b.$$

Construct P'' as follows.

$$\begin{array}{c} P'' \\ \hline b \\ a \\ \vdots \end{array} \qquad P_i(1) = b, \text{ and } P_i(2) = a \forall i \in N. \tag{19.6}$$

From the definition of monotonicity, we see that

- for $P \to P''$, we see that $f(P) = a$ and a's relative position gets weakly better from P to P'' for all $i \in N$, then by monotonicity $f(P'') = a$.
- similarly for $P' \to P''$, we see that $f(P') = b$ and b's relative position gets weakly better from P' to P'' for all $i \in N$, by monotonicity $f(P'') = b$. This is a contradiction.

■

The conclusions of this lecture is therefore captured in Figures 19.1 and 19.2 – the properties discusses coincides under strategyproofness.

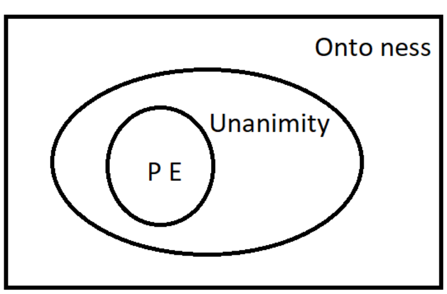

Figure 19.1: Representation of all the properties of Social choice function

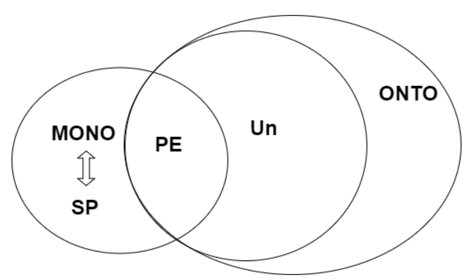

Figure 19.2: The big picture of all axioms of social choice setting