CS698W: Game Theory and Collective Choice **Jul-Nov 2017**

Lecture 18: September 12, 2017

Lecturer: Swaprava Nath *Scribe(s): Kunal Chaturvedi*

Disclaimer: *These notes aggregate content from several texts and have not been subjected to the usual scrutiny deserved by formal publications. If you find errors, please bring to the notice of the Instructor at swaprava@cse.iitk.ac.in.*

18.1 Recap

In the last class, we discussed about a key property for social welfare functions, namely independence of irrelevant alternatives (IIA) and saw that scoring rules does not satisfy it. We stated the celebrated Arrow's impossibility theorem.

The proof of Arrow's theorem is via two lemmas. We proved the *field expansion lemma* in the previous class. In this class we will prove the *group contraction lemma.*

18.2 Continuing the proof of Arrow's theorem

Theorem 18.1 (Arrow 1950) *For $|A| \geqslant 3$, if an ASWF F satisfies* weak Pareto *and* IIA *then it must be dictatorial.*

Proof:[(Contd.)] To complete the proof we state and prove the *group contraction lemma*

Lemma 18.2 (Group Contraction) *Let the group $G \subseteq N, G \neq \emptyset$ be decisive. Then $\exists G' \subset G$ which is also decisive.*

Proof: If $|G| = 1$, the lemma trivially holds. For $|G| \geq 2$, consider two subsets of G, namely $G_1 \subset G$ and $G_2 = G \setminus G_1$. Construct a preference profile R as follows.

G_1	G_2	$N \setminus G$
a	c	b
b	a	c
c	b	a

Note that for all $i \in G$, aP_ib. Since G is decisive, we get

$$a\hat{F}(R)b. \tag{18.1}$$

Consider two possible cases for the social ordering F between alternatives a and c.

Case 1: $a\hat{F}(R)c$.

Consider G_1. We see that, by construction

$$aP_ic,\ \forall i \in G_1,\ \text{and}\ cP_ja,\ \forall j \notin G_1.$$

Consider any arbitrary preference ordering R' where the above condition holds. Since, F satisfies IIA, we conclude that $a\hat{F}(R)c$. Hence $\bar{D}_{G_1}(a,c)$. Using field expansion lemma, we get that G_1 is decisive.

Case 2: $\neg\ (a\hat{F}(R)c) \Rightarrow cF(R)a$

Also from Equation 18.1, we have $a\hat{F}(R)b$. Therefore, by transitivity, $c\hat{F}(R)b$.

Consider G_2. By construction

$$cP_ib,\ \forall i \in G_2,\ \text{and}\ bP_jc,\ \forall j \notin G_2.$$

Consider again any arbitrary preference ordering R' where the above condition holds. Since, F satisfies IIA, we conclude that $c\hat{F}(R)b$. Hence $\bar{D}_{G_2}(c,b)$. Using field expansion lemma, we get that G_2 is decisive. ■

Let us finish the proof of Arrow's theorem. By weak Pareto, N is decisive. By Lemma 18.2, $\exists i \in N$ such that $\{i\}$ is decisive. As we have a singleton set which is decisive, we conclude that i is the dictator. ■

Observation: For a given F, the dictator i is unique.

18.3 Social Choice Setting

Arrow's theorem tells us that we cannot hope to find a meaningful aggregation of preferences that satisfies some very basic desired properties. There are restrictions put on the preferences, e.g., single peaked preferences, where there are non-dictatorial results.

However, we will consider a different path of relaxing the conditions. The ASWF setting asks for a social ordering which may be too much to satisfy. An alternative way to formulate the aggregation problem is to consider the setting of "social choice" functions where the outcome is a single alternative instead of a ranking. Hence a *social choice function* is a map $f : \mathcal{P}^n \mapsto A$, where $\mathcal{P}$ is the set of linear orders, i.e., strict preferences. The set A represents the set of alternatives.

A representative case of this kind of social setting is voting.

18.3.1 Examples of Voting Protocols

This is a list of some voting protocols.

1. **Plurality**: Every voter votes for his/her most favorite candidate and the candidate with highest number of votes is the winner. **Example(s):** Voting system in India, USA, Britain, Canada etc.

2. **Plurality with runoff (two stages)**: In this case the top two candidates from the first round of voting advance to the second round of voting. In the second round, the highest voted candidate wins overall. **Example(s):** French presidential election, Rajya Sabha election in India.

3. **Approval Voting**: Each voter casts a single vote for as many candidates as he wants. The candidate with the most votes is the winner. **Example(s):** Approval rating is used by the Mathematical Association of America.

4. **Scoring Rule**: Scores are assigned to candidates according to the vector $(s_1, s_2, \ldots, s_m)$. The highest scoring candidate wins. Some scoring rules which we have seen in the previous lecture are.

   (a) Borda Count

   (b) Veto Rule

(c) Plurality is a special case of this class.

We calculate scores as we calculated before but now we only consider the highest scorer as the winner.

5. **Maximin**: Candidates are visualized as vertices in a directed graph and are considered pairwise to assign points. Edges point from the winner to the loser in pairwise runoff. Edges are given weights according to the pairwise winning margin. The candidate with the largest margin of weights wins.

6. **Copeland**: Candidate with maximum pairwise wins is the winner (unlike maximin, here the edge weights do not matter).

None of the voting rules is Pareto superior than another. Given some social objective for the social planner, different voting rules perform better or worse. Every voting rule has some merits, and therefore they survived. However, in future lectures, we will consider a general set of preferences where none of these voting rules perform well.

Condorcet Paradox: Pair-wise runoffs can lead to paradoxical situations. Suppose we have three candidates a, b, c and three voters whose preferences are as follows.

Voter 1	Voter 2	Voter 3
a	c	b
b	a	c
c	b	a

In this case no candidate beats everyone else in pairwise elections.

Definition 18.3 (Condorcet Consistency) *If there exists a candidate which is preferred over every candidate in pairwise runoffs, then he* should *be the winner.*

Note: Condorcet consistency does not hold for all voting schemes. Example : scoring rules.