CS698W: Game Theory and Collective Choice **Jul-Nov 2017**

Lecture 17: September 8, 2017

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17.1 Recap

We defined the Arrovian social welfare function (ASWF) to be a mapping from the set of all preference profiles of n agents to a single preference profile. Hence it is a function $F : \mathcal{R}^n \rightarrow \mathcal{R}$, where $\mathcal{R}$ is the set of all possible orderings over $|A|$ candidates. Two desirable properties that were listed were

Definition 17.1 *A Social Welfare Function F satisfies* weak Pareto *(WP) if*

$$\forall a, b \in A, \quad [aP_i b, \ \forall i \in N] \implies [a\hat{F}(R)b].$$

Definition 17.2 *A Social Welfare Function F satisfies* strong Pareto *if*

$$\forall a, b \in A, \quad [aR_i b, \ \forall i \in N, \ \exists j, \ aP_j b] \implies [a\hat{F}(R)b].$$

Clearly, strong Pareto $\implies$ weak Pareto. The other desirable property in the ASWF setup is *indepedence of irrelevant alternatives.*

17.2 Independence of Irrelevant Alternatives

This property is the crux of Arrow's impossibility result. It is a property that connects two different preference profiles.

Two preferences of player i, say R_i and $R'_i \in \mathcal{R}$ are said to *agree* over $\{a, b\}$ if for agent i

- $aP_i b \Leftrightarrow aP'_i b$
- $bP_i a \Leftrightarrow bP'_i a$
- $aI_i b \Leftrightarrow aI'_i b$

We denote this using the notation $R_i|_{a,b} = R'_i|_{a,b}$. Two preference profiles R, R' *agree* if for every $i \in N$, $R_i|_{a,b} = R'_i|_{a,b}$ and is denoted by

$$R|_{a,b} = R'|_{a,b}\,.$$

Definition 17.3 (Independence of Irrelevant Alternatives) *An ASWF F satisfies* indepedence of irrelevant alternatives (IIA) *if for all $a, b \in A$*

$$[R|_{a,b} = R'|_{a,b}] \implies [F(R)|_{a,b} = F(R')|_{a,b}].$$

Illustration Consider an ASWF F, where given the position of the ranking for every agent, some scores are assigned to the candidates. Formally, say the score vector is $(s_1, s_2, s_3, \ldots, s_m), \quad s_i \geq s_{i+1}, i = 1, 2, \ldots, m-1, s_i \geq 0, \forall i \in N$. Finally all scores of a particular candidate are added and the final ranking is based on the decreasing order of these scores. This is one special class of ASWF.

Some well-known scoring rules are described below:

- **Plurality**: In this case we assing top score, i.e 1 to s_1 and 0 to all others, So $s_1 = 1$,and $s_2 = s_3 = \ldots = s_m = 0$.

  Question: Does plurality satisfy IIA?

  Consider two preference profiles R and R'. The preferences of 4 voters are as follows.

R					R'			
a	a	c	d		d	c	b	b
b	c	b	c		a	a	c	a
c	b	a	b		b	b	a	d
d	d	d	a		c	d	d	c

  Plurality gives a social ordering between a and b as:

  $$a\hat{F}^{\mathrm{Plu}}(R)b, \quad \text{and} \quad b\hat{F}^{\mathrm{Plu}}(R')a.$$

  However, we see that the ordering of a and b remains same for every agent in R and R'. IIA would require that the social ordering remain unchanged, which does not happen for plurality. Thus we conclude that plurality does not satisfy *IIA*.

- **Borda**: The scoring rule in this case is: $s_1 = m - 1, s_2 = m - 2, \ldots, s_{m-1} = 1, s_m = 0$.

- **Veto**: The scoring rule is: $s_1 = s_2 = \ldots = s_{m-1} = 1, s_m = 0$. We can check by suitable examples that neither Borda nor veto satisfies *IIA*.

- **Dictatorial**: A voting rule is dictatorial if it always selects the preference ordering of a distinguished agent, whom we call the *dictator*. Thus it is trivial that a dictatorial voting rule satisfies *IIA*.

We are now going to present a classic result in social choice.

Theorem 17.4 (Arrow 1950) *For $|A| \geqslant 3$, if an ASWF F satisfies* weak Pareto *and* IIA *then it must be dictatorial.*

Proof: The proof of the following two lemmas will lead us to eventually prove Arrow's theorem. Informally we state the basic statements of the lemmas as follows.

1. *Field Expansion Lemma*: if a group $G \subseteq N, G \neq \emptyset$ is *decisive* over a, b, then it is decisive over all pairs of alternatives. Informally, a decisive group is a group such that if every agent in that group agrees on a ranking between a pair of alternatives, that ranking is reflected in the social ranking. Therefore, with this lemma, it is enough to call a group *decisive* since it implies that it is decisive over all pairs of alternatives.

2. *Group Contraction Lemma*: if a group G is decisive, there exists a strict subset of G that is also decisive.

First we define decisiveness formally.

Definition 17.5 *Given $F : \mathcal{R}^n \to \mathcal{R}$. Let $G \subseteq N, G \neq \emptyset$.*

1. *G is* almost decisive *over a, b if*

$$[aP_ib,\ \forall i \in G,\ \textit{and}\ bP_ja, \forall j \notin G] \implies [a\hat{F}(R)b]$$

2. *G is called* decisive *over a, b if*

$$[aP_ib,\ \forall i \in G] \implies [a\hat{F}(R)b]$$

We will use the notation $\bar{D}_G(a, b)$ to denote that G is almost decisive over a, b and $D_G(a, b)$ to denote that G is decisive over a, b. Clearly, $D_G(a, b) \implies \bar{D}_G(a, b)$.

Lemma 17.6 (Field Expansion) *Let F satisfies* weak Pareto *and* IIA *then $\forall a, b, x, y, a \neq b, x \neq y$, we have*

$$\bar{D}_G(a, b) \implies D_G(x, y).$$

Proof: We consider the following set of exhaustive cases to prove this lemma.

1. $\bar{D}_G(a, b) \implies D_G(a, y)$ where $y \neq a, b$
2. $\bar{D}_G(a, b) \implies D_G(x, b)$ where $x \neq a, b$
3. $\bar{D}_G(a, b) \implies D_G(x, y)$ where $x \neq a, b$ and $y \neq a, b$
4. $\bar{D}_G(a, b) \implies D_G(x, a)$ where $x \neq a, b$
5. $\bar{D}_G(a, b) \implies D_G(b, y)$ where $y \neq a, b$
6. $\bar{D}_G(a, b) \implies D_G(b, a)$
7. $\bar{D}_G(a, b) \implies D_G(a, b)$

***Case 1*:** Given: $\bar{D}_G(a, b)$, we need to show $D_G(a, y)$. Pick arbitrary R such that,

$$aP_iy,\ \forall i \in G,\quad \text{need to show } a\hat{F}(R)y.$$

Construct R' as follows.

G	$N \setminus G$
$a \succ b \succ y$	$b \succ a$ and $b \succ y$

Where $a \succ b$ denotes a is more preferred than b. For the agents in $N \setminus G$, we ensure that the ranking of a and y remain identical to the ranking of these two alternatives in R. Therefore we have

$$R|_{a,y} = R'|_{a,y}.$$

Now since $aR'_ib, \forall i \in G$ and $bR'_ja, \forall j \notin G$, by definition of $\bar{D}_G(a, b)$ we conclude that $a\hat{F}(R')b$. Since b is preferred over y by all agents in N, WP implies that $b\hat{F}(R')y$. Using transitivity of $F(R')$,we have, $a\hat{F}(R')y$. Since the relative ranking of a and y in R and R' are same, using IIA we get $a\hat{F}(R)y$.

***Case 2*:** Given: $\bar{D}_G(a, b)$, we need to show $D_G(x, b)$. Pick arbitrary R such that,

$$xP_ib,\ \forall i \in G,\quad \text{need to show } x\hat{F}(R)b.$$

Construct R' as follows.

G	$N \setminus G$
$x \succ a \succ b$	$x \succ a$ and $b \succ a$

For the agents in $N \setminus G$, we ensure that the ranking of x and b remain identical to the ranking of these two alternatives in R. Therefore we have

$$R|_{x,b} = R'|_{x,b}\,.$$

Now since $aR'_ib, \forall i \in G$ and $bR'_ja, \forall j \notin G$, by definition of $\bar{D}_G(a,b)$ we conclude that $a\hat{F}(R')b$. Since x is preferred over a by all agents in N, WP implies that $x\hat{F}(R')a$. Using transitivity of $F(R')$,we have, $x\hat{F}(R')b$. Since the relative ranking of a and y in R and R' are same, using IIA we get $x\hat{F}(R)b$.

***Case 3*:**

$$\begin{aligned}\bar{D}_G(a,b) &\implies D_G(a,y),\ y \neq a,b && \text{(by Case 1)}\\ &\implies \bar{D}_G(a,y) && \text{(by definition)}\\ &\implies \bar{D}_G(x,y),\ \text{as } x \neq a,y && \text{(by Case 2)}\end{aligned}$$

***Case 4*:**

$$\begin{aligned}\bar{D}_G(a,b) &\implies D_G(x,b),\ x \neq a,b && \text{(by Case 2)}\\ &\implies \bar{D}_G(x,b) && \text{(by definition)}\\ &\implies \bar{D}_G(x,a),\ \text{as } a \neq b,x && \text{(by Case 1)}\end{aligned}$$

***Case 5*:**

$$\begin{aligned}\bar{D}_G(a,b) &\implies D_G(a,y),\ y \neq a,b && \text{(by Case 1)}\\ &\implies \bar{D}_G(a,y) && \text{(by definition)}\\ &\implies \bar{D}_G(b,y),\ \text{as } b \neq a,y && \text{(by Case 2)}\end{aligned}$$

***Case 6*:**

$$\begin{aligned}\bar{D}_G(a,b) &\implies D_G(x,b),\ x \neq a,b && \text{(by Case 2)}\\ &\implies \bar{D}_G(x,b) && \text{(by definition)}\\ &\implies \bar{D}_G(a,b),\ \text{as } a \neq b,x && \text{(by Case 2)}\end{aligned}$$

***Case 7*:**

$$\begin{aligned}\bar{D}_G(a,b) &\implies D_G(b,y),\ y \neq a,b && \text{(by Case 5)}\\ &\implies \bar{D}_G(b,y) && \text{(by definition)}\\ &\implies \bar{D}_G(b,a),\ \text{as } a \neq b,y && \text{(by Case 1)}\end{aligned}$$

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In the next class we will prove the *Group Contraction Lemma* to complete our proof of *Arrow's Theorem*.

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