

## Lecture 15: 5th September, 2017

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## 15.1 Recap of Mechanism Design Basics

In the previous class, we discussed that in mechanism design we look at the game from the designer's perspective. Our motive is to design games such that they follow certain properties in their equilibria. Typical examples of where mechanisms are used include auctions, voting and matchings. Notice that in each of these settings, the players have private information. Formally, we model the setting as follows:

### Setup

- $N = \{1, 2, \dots, n\}$  – a set of *players*
- $X$  – a set of *outcomes*
- $\Theta_i$  – set of *types* of player  $i$ . This is used to model the private information of player  $i$ .
- $u_i : X \times \Theta_i \rightarrow \mathbb{R}$  – the *utility* that player  $i$  gets. This depends on the outcome and also his true type (private value model).

## 15.2 Social Choice Function

Mechanism design starts with an objective of the *social planner* or *mechanism designer*. The objective could be, e.g., to allocate an object to the individual who values it the most. However, the values of the agents are their private information or types. Therefore, this objective can be represented by a function that maps the type space of all the agents to the outcome space. Formally, this function, which we call the *social choice function* (SCF), is defined as  $f : \Theta \rightarrow X$ , where  $\Theta = \Theta_1 \times \Theta_2 \times \dots \times \Theta_n$ . That is, given a type profile  $\theta = (\theta_1, \theta_2, \dots, \theta_n)$ , the outcome is given by  $f(\theta)$ .

### 15.2.1 Examples of Social Choice Functions

Since SCFs reflect the objective of the social planner, it can be very generic. We will later see that the function can be complex depending on the domain of the problems. However, some commonly used SCFs are described below.

1. *Utilitarian*: A SCF  $f : \Theta \rightarrow X$  is *utilitarian* if it *always* chooses the outcome that maximizes the sum of utilities of each player. Formally,

$$f(\theta_i, \theta_{-i}) = \arg \max_{a \in X} \sum_{i \in N} u_i(a, \theta_i).$$

In the above,  $\sum_{i \in N} u_i(a, \theta_i)$  is also called the *social welfare* of the outcome  $a$ . Therefore, an utilitarian SCF chooses the outcome that maximizes the social welfare.

2. *Egalitarian*: A SCF  $f : \Theta \rightarrow X$  is *egalitarian* if it *always* chooses the outcome that maximizes the minimum utility that any player gets on that outcome. Formally,

$$f(\theta_i, \theta_{-i}) = \arg \max_{a \in X} \left[ \min_{i \in N} u_i(a, \theta_i) \right]$$

Note that in an egalitarian SCF, we look at minimum utility that any player gets under an outcome and then we choose the outcome that maximizes this. This SCF, in some sense, aims to equalize the utilities of the agents.

## 15.3 Mechanisms

**Definition 15.1 (Mechanism)** A Mechanism  $\mathcal{M}$  is a collection of message spaces,  $(M_1, M_2, \dots, M_n)$  and a decision rule  $g : \times_{i \in N} M_i \rightarrow X$ . It is typically denoted as  $\mathcal{M} = (M_1, M_2, \dots, M_n, g)$  or  $\mathcal{M} = \langle M, g \rangle$ , where  $M = \times_{i \in N} M_i$ .

The idea of message spaces has been kept abstract on purpose to allow the model to be generic. One can think of each message space as a set of actions that a player can take. Players communicate with the central authority using messages and these messages are used by the decision function  $g$  to decide an outcome. There is a special case in which the central authority simply asks the players to report their types. Such a mechanism is called a *direct mechanism* and is defined as follows.

**Definition 15.2 (Direct Mechanism)** A mechanism  $\mathcal{M} = \langle M, g \rangle$  is a direct mechanism if  $M_i = \Theta_i, \forall i \in N$  and  $g$  is the SCF  $f$ . Therefore,  $\langle \Theta, f \rangle$  is a direct mechanism.

Typically, the message that a player sends to the central authority depends on the player's type so we can also think of *message mapping*  $m_i : \Theta_i \rightarrow M_i$  as a strategy. Therefore, a player's action depends on the type that player receives. Suppose player  $i$  has received a type  $\theta_i \in \Theta_i$ . We now give the following definition:

**Definition 15.3 (Weakly Dominant Message)** Let  $\mathcal{M} = \langle M, g \rangle$  be a mechanism. A message  $m_i \in M_i$  is weakly dominant for player  $i$  at  $\theta_i$  if

$$u_i(g(m_i, m_{-i}), \theta_i) \geq u_i(g(m'_i, m_{-i}), \theta_i), \quad \forall m'_i \in M_i, \forall m_{-i} \in M_{-i}.$$

Note that we define a message  $m_i \in M_i$  as a weakly dominant *for a player  $i$  when his type is  $\theta_i$* . A weakly dominant message is optimal for player  $i$  (given  $\theta_i$ ) irrespective of what the other players communicate.

**Definition 15.4 (Implemented in Dominant Strategies)** An SCF  $f : \Theta \rightarrow X$  is implemented in dominant strategies by a mechanism  $\mathcal{M} = \langle M, g \rangle$  if the following two conditions hold:

1.  $\forall i \in N, \exists m_i : \Theta_i \rightarrow M_i$  such that  $\forall \theta_i \in \Theta_i, m_i(\theta_i)$  is weakly dominant for player  $i$  at  $\theta_i$ , and
2.  $g(m_i(\theta_i), m_{-i}(\theta_{-i})) = f(\theta_i, \theta_{-i}), \forall \theta \in \Theta$ .

In the above case, we also say that  $f$  is *dominant strategy implementable* (DSI) and  $\mathcal{M} = \langle M, g \rangle$  implements  $f$ .

**Definition 15.5 (Strategyproof or Dominant Strategy Incentive Compatible)** A direct mechanism  $\langle \Theta, f \rangle$  is strategyproof or dominant strategy incentive compatible (DSIC) if

$$u_i(f(\theta_i, \hat{\theta}_{-i}), \theta_i) \geq u_i(f(\theta'_i, \hat{\theta}_{-i}), \theta_i), \forall \theta_i, \theta'_i \in \Theta_i, \forall \hat{\theta}_{-i} \in \Theta_{-i}, \forall i \in N.$$

Strategyproof says that no matter what the other players do (i.e., whether or not they reveal their types truthfully), revealing your type truthfully is at least as good as every other strategy (recall that a strategy is a message mapping). Formally,  $m_i(\theta_i) = \theta_i$  is a weakly dominant message for player  $i, \forall \theta_i \in \Theta_i, \forall i \in N$ .

## 15.4 Revelation Principle for DSI SCFs

**Theorem 15.6** *If there exists an indirect mechanism that implements a SCF  $f$  in dominant strategies, then there exists a direct mechanism to implement  $f$  (i.e.,  $f$  is DSIC).*

**Proof:** Let  $f$  be implemented in dominant strategies by the indirect mechanism  $\mathcal{M} = \langle M, g \rangle$ . Hence  $\forall i \in N, \exists m_i : \Theta_i \rightarrow M_i$  such that  $m_i(\theta_i)$  is weakly dominant for player  $i$  at  $\theta_i, \forall \theta_i \in \Theta_i$ . Hence  $\forall i \in N$

$$u_i(g(m_i(\theta_i), m''_{-i}), \theta_i) \geq u_i(g(m', m''_{-i}), \theta_i), \forall m' \in M_i, \forall m''_{-i} \in M_{-i}, \forall \theta \in \Theta.$$

And

$$g(m_i(\theta_i), m_{-i}(\theta_{-i})) = f(\theta_i, \theta_{-i}), \forall \theta \in \Theta.$$

Since the above equations hold for all  $\forall m' \in M_i, \forall m_{-i} \in M_{-i}, \forall \theta \in \Theta$ , they must also hold when  $m' = m_i(\theta'_i)$  and  $m''_{-i} = m_{-i}(\theta_{-i})$ . Plugging this into the equation, we get

$$u_i\left(g(m_i(\theta_i), m_{-i}(\theta_{-i})), \theta_i\right) \geq u_i\left(g(m_i(\theta'_i), m_{-i}(\theta_{-i})), \theta_i\right), \forall \theta'_i \in \Theta_i, \forall \theta \in \Theta, \forall i \in N.$$

Note that  $g(m_i(\theta_i), m_{-i}(\theta_{-i})) = f(\theta_i, \theta_{-i})$  and  $g(m_i(\theta'_i), m_{-i}(\theta_{-i})) = f(\theta'_i, \theta_{-i})$  since  $f$  is DSI. This gives  $\forall i \in N$

$$u_i(f(\theta_i, \theta_{-i})) \geq u_i(f(\theta'_i, \theta_{-i})), \forall \theta'_i \in \Theta_i, \forall \theta \in \Theta.$$

This is precisely the definition of strategyproof social choice functions. Therefore,  $f$  is DSIC. ■

## 15.5 Summary

This lecture formalized the setting from a mechanism designer's perspective by defining a social choice function. The SCF determines the outcome that is taken and is a function of the players' reported types. We then defined indirect mechanisms, where players communicate with the central authority via messages and the central authority determines the outcome using a decision rule ( $g$ ). A special case of this is when the central authority asks players to report their types and the decision rule is simply the SCF ( $f$ ). We then

defined weakly dominant messages as those which maximize player  $i$ 's utility when his type is  $\theta_i$ , irrespective of how the other players play. We later defined what it means for a SCF to be implemented in dominant strategies and talked about strategyproof SCFs or dominant strategy incentive compatible (DSIC) SCFs. We concluded by stating and proving the Revelation Theorem which says that if  $f$  is implemented by an indirect mechanism in dominant strategies, then there is also a direct mechanism that implements  $f$ .