CS698W: Game Theory and Collective Choice	**Jul-Nov 2017**

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Disclaimer: *These notes aggregate content from several texts and have not been subjected to the usual scrutiny deserved by formal publications. If you find errors, please bring to the notice of the Instructor at swaprava@cse.iitk.ac.in.*

14.1 Examples of Bayesian equilibrium

14.1.1 Sealed Bid Auction

In this game, we have a seller (who is not a player) willing to sell a commodity via an auction and two buyers (these are the competing players) who place sealed bids (secret to each other) on the commodity.

Values $\in [0,1]$: these are the types $\theta_i,\ i \in \{1,2\}$.

Bids $\in [0,1]$: these are the actions b_i.

The probability distribution over this continuous range is given by:

$f_1(\theta_2 \mid \theta_1) = f_1(\theta_2) = 1, \quad \theta_2 \in [0,1]$.

$f_2(\theta_1 \mid \theta_2) = f_2(\theta_1) = 1, \quad \theta_1 \in [0,1]$.

The above two are consistent with the common prior: $f(\theta_1, \theta_2) = 1,\ (\theta_1, \theta_2) \in [0,1]^2$

14.1.1.1 First Price Auction

Highest bidder is winner, who then has to pay the bid.

Utilities :

$u_1(b_1, b_2, \theta_1, \theta_2) = (\theta_1 - b_1)\ \mathrm{I}\{b_1 \geq b_2\}$

$u_2(b_1, b_2, \theta_1, \theta_2) = (\theta_2 - b_2)\ \mathrm{I}\{b_1 < b_2\}$

Say $b_1 = s_1(\theta_1) = \alpha_1\theta_1$ and $b_2 = s_2(\theta_2) = \alpha_2\theta_2$ (Assuming bid to be a fraction of true valuation) where s_1, s_2 are respective strategies.

Player 1's problem :

$$\begin{aligned}
&\max_{\sigma_1} \mathbb{E}[U_1(\sigma_1, \sigma_2^* | \theta_1) \\
&= \max_{b_1 \in [0,\alpha_2]} \int_0^1 f(\theta_2)(\theta_1 - b_1)\ \mathrm{I}\{b_1 \geq \alpha_2\theta_2\} d\theta_2 \\
&= \max_{b_1 \in [0,\alpha_2]} (\theta_1 - b_1) \cdot \frac{b_1}{\alpha_2}
\end{aligned}$$

Differentiating w.r.t b_1 to maximize, and using $b_1 \in [0, \alpha_2]$, we get

$$b_1 = \min\{\theta_1/2, \alpha_2\}$$

Similarly, $b_2 = \min\{\theta_2/2, \alpha_1\}$.

Thus, $((\frac{\theta_1}{2}, \frac{\theta_2}{2}), \text{uniform prior})$ is a Bayesian equilibrium.

14.1.1.2 Second Price Auction

The player who has the highest bid wins and pays the second highest bid.

Utilities :

$$u_1(b_1, b_2, \theta_1, \theta_2) = (\theta_1 - b_2)\ \mathrm{I}\{b_1 \geq b_2\}$$

$$u_2(b_1, b_2, \theta_1, \theta_2) = (\theta_2 - b_1)\ \mathrm{I}\{b_1 < b_2\}$$

Player 1's problem :

$$\begin{aligned}
&\max_{\sigma_1} \mathbb{E}[U_1(\sigma_1, \sigma_2^*|\theta_1)] \\
&= \max_{b_1 \in [0,\alpha_2]} \int_0^1 f(\theta_2)(\theta_1 - \alpha_2\theta_2)\ \mathrm{I}\{b_1 \geq \alpha_2\theta_2\} d\theta_2 \\
&= \max_{b_1 \in [0,\alpha_2]} \int_0^{b_1/\alpha_2} (\theta_1 - \alpha_2\theta_2) d\theta_2 \\
&= \max_{b_1 \in [0,\alpha_2]} \theta_1.b_1/\alpha_2 - \alpha_2/2.b_1^2/\alpha_2^2
\end{aligned}$$

Differentiating w.r.t b_1 to maximize, we get

$$b_1 = \theta_1, \text{ and similarly for player 2, } b_2 = \theta_2.$$

Thus, $((\theta_1, \theta_2), \text{uniform prior})$ is a Bayesian equilibrium.

Arbitrary prior: For a non-uniform prior, we consider the same maximization problem for player 1.

$$\begin{aligned}
&\max_{b_1 \in [0,\alpha_2]} \int_0^{b_1/\alpha_2} f(\theta_2)(\theta_1 - \alpha_2\theta_2) d\theta_2 \\
&= \max_{b_1 \in [0,\alpha_2]} \theta_1.F(b_1/\alpha_2) - \alpha_2 \int_0^{b_1/\alpha_2} \theta_2 f(\theta_2) d\theta_2 \\
&= \max_{b_1 \in [0,\alpha_2]} \theta_1.F(b_1/\alpha_2) - b_1.F(b_1/\alpha_2) + \alpha_2 \int_0^{b_1/\alpha_2} F(\theta_2) d\theta_2 \qquad \text{(integrating by parts)}
\end{aligned}$$

Differentiating w.r.t b_1 to maximize, we get

$$\theta_1.f(b_1/\alpha_2)/\alpha_2 - F(b_1/\alpha_2) - b_1.f(b_1/\alpha_2)/\alpha_2 + F(b_1/\alpha_2) = 0.$$

Thus we get

$$b_1 = \theta_1, \text{ and similarly for player 2, } b_2 = \theta_2.$$

This condition is the same we got with uniform prior. Hence, every prior will have an equilibrium where the bid is to reveal the true type. Second price auction is therefore called a prior free auction.

14.2 Mechanism Design

In Game Theory, we take an agent's approach and the guarantees are predictive.

In Mechanism Design, we take a designer's approach and the guarantees are prescriptive.

Some examples are:

- Matching (student - university so that nobody breaks their current allocation)
- Auction (Combinatorial)
- Spectrum, IPL
- Voting

14.2.1 Setup

N	$= \{1, 2, ..., n\}$	Set of players/agents
X		Set of outcomes
Θ_i		Set of types of i (Private information of i)
u_i	$: X \times \Theta_i \to \mathbb{R}$	Private value model (One's utility dependent only on his type, after fixing the outcome)
u_i	$: X \times \Theta \to \mathbb{R}$	Interdependent value model (Utility depends on everyone's types, after fixing the outcome)

14.2.2 Examples

Voting $X = \{a, b, c, \ldots\}$ – Set of Outcomes – the set of candidates.

θ_i is a linear order over the candidates.

e.g., Let $\theta_1 = a \succ b \succ c$.

v_i is any vNM utility which is consistent with θ_i.

$\Rightarrow v_1(a) > v_1(b) > v_1(c) \qquad u_1(a, v_1) = v_1(a)$.

Single Object Allocation $x \in X$ is a tuple (a, p) – allocation and payment.

$p_i \in \mathbb{R}$ (price charged).

$a = (a_1, a_2, ..., a_n)$ (Whom to allocate).

$a_i \in \{0, 1\} \qquad \Sigma_i a_i \leq 1 \qquad$ (not given to more than one person).

$\theta_i \in \mathbb{R}$ (Satisfaction if the object is obtained by i).

$u_i(x, \theta_i) = u_i((a, p), \theta_i) = a_i\theta_i - p_i$ quasi-linear payoff/utility (Linear in terms of payment).

Public Project $x = (a, p)$, $x \in X$ is a choice of a project a and tax assigned p.

$\theta_i : A \to \mathbb{R} \quad \theta_i \in \mathbb{R}^{|A|}$.

$u_i(x, \theta_i) = \theta_i(a) - p_i$.