CS698W: Game Theory and Collective Choice **Jul-Nov 2017**

Lecture 13: August 30, 2017

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13.1 Recap

In the previous lecture, for Bayesian games, two different types of utilities were discussed: ex-ante utility and ex-interim utility. Ex-ante utility is the utility of any player before observing own type of profile and is expressed as

$$U_i(\sigma_i, \sigma_{-i}) = \sum_{\theta \in \Theta} P(\theta) U_i(\sigma(\theta), \theta).$$

Where, $\sigma(\theta) = (\sigma_1(\theta_1), \sigma_2(\theta_2), \ldots, \sigma_n(\theta_n))$. Hence

$$U_i(\sigma_i, \sigma_{-i}) = \sum_{\theta \in \Theta} P(\theta) \sum_{(a1,a2,\ldots,a_n) \in A} \left(\prod_{j \in N} \sigma_j(\theta_j, a_j) \right) u(a_1, \ldots, a_n, \theta_1, \ldots, \theta_n). \tag{13.1}$$

And, while calculating ex-intrim utility the player knows own type of profile and is expressed as:

$$U_i(\sigma | \theta_i) = \sum_{\theta_{-i} \in \Theta_i} P(\theta_{-i} | \theta_i) U_i(\sigma(\theta), \theta) \tag{13.2}$$

The relation between the two utilities is expressed as

$$U_i(\sigma_i, \sigma_{-i}) = \sum_{\theta_i \in \Theta_i} P(\theta_i) U_i(\sigma | \theta_i). \tag{13.3}$$

13.2 Equilibrium Concepts

Definition 13.1 (Nash Equilibrium) *In a Bayesian game with prior P, (σ^*, P) is a Nash equilibrium if*

$$U_i(\sigma_i^*, \sigma_{-i}^*) \geqslant U_i(\sigma_i', \sigma_{-i}^*), \ \forall \sigma_i', \forall i \in N \tag{13.4}$$

Therefore, for player i, playing σ_i^* is a best response if other players play σ_{-i}^* before observing her own type.

Definition 13.2 (Bayesian Equilibrium) *In a Bayesian game with prior P, (σ^*, P) is a Bayesian equilibrium if*

$$U_i(\sigma_i^*, \sigma_{-i}^* | \theta_i) \geqslant U_i(\sigma_i', \sigma_{-i}^* | \theta_i), \ \forall \theta_i \in \Theta_i, \forall \sigma_i', \forall i \in N \tag{13.5}$$

Therefore, for player i, playing σ_i^* is a best response if other players play σ_{-i}^* after observing her own type.

Observe that σ_i' in Equation 13.5 can be replaced with pure actions WLOG, i.e. Therefore,

$$U_i(\sigma_i^*, \sigma_{-i}^*|\theta_i) \geqslant U_i(a_i, \sigma_{-i}^*|\theta_i),\ \forall \theta_i \in \Theta_i, \forall a_i \in A_i, \forall i \in N. \tag{13.6}$$

This is because if the inequality holds for every pure action $a_i \in A_i$, then it must hold even when such actions are mixed probabilistically.

13.3 Equivalence of the two equilibrium concepts

Theorem 13.3 *In finite Bayesian games (σ^*, P) is a Bayesian equilibrium iff it is a Nash equilibrium.*

Proof: ($\Rightarrow$): Suppose (σ^*, P) is BE. Then

$$U_i(\sigma_i^*, \sigma_{-i}^*|\theta_i) \geq U_i(\sigma_i', \sigma_{-i}^*|\theta_i),\ \forall \sigma_i', \forall i \in N, \forall \theta_i \in \Theta_i \tag{13.7}$$

Now, the ex-ante utility of player i at $(\sigma_i^*, \sigma_{-i}^*)$ is

$$\begin{aligned}
U_i(\sigma_i^*, \sigma_{-i}^*) &= \sum_{\theta_i \in \Theta_i} P(\theta_i) U_i(\sigma_i^*, \sigma_{-i}^*|\theta_i) && \text{(from Eqn. 13.3)}\\
&\geqslant \sum_{\theta_i \in \Theta_i} P(\theta_i) U_i(\sigma_i', \sigma_{-i}^*|\theta_i) && \text{(from Eqn. 13.7)}\\
&= U_i(\sigma_i', \sigma_{-i}^*).
\end{aligned}$$

Hence $((\sigma^*, P))$ is a Nash equilibrium.

($\Leftarrow$): Suppose (σ^*, P) is a Nash equilibrium. Assume for contradiction that (σ^*, P) is not a Bayesian equilibrium.

Then, $\exists\ a_i \in A_i$, some $\theta_i \in \Theta_i$, some $i \in N$ such that,

$$U_i(a_i, \sigma_{-i}^*|\theta_i) > U_i(\sigma_i^*, \sigma_{-i}^*|\theta_i). \tag{13.8}$$

Consider the following strategy $\hat{\sigma}_i$ of i

$$\begin{aligned}
\hat{\sigma}_i(\theta_i') &\equiv \sigma_i^*(\theta_i'),\ \forall \theta_i' \in \Theta_i \setminus \{\theta_i\},\\
\hat{\sigma}_i(\theta_i, a_i) &= 1\\
\hat{\sigma}_i(\theta_i, b_i) &= 0,\ \forall b_i \in A_i \setminus \{a_i\}.
\end{aligned}$$

Hence, the ex-ante utility of player i at $(\hat{\sigma}_i, \sigma_{-i}^*)$ is

$$\begin{aligned}
U_i(\hat{\sigma}_i, \sigma_{-i}^*) &= \sum_{\tilde{\theta}_i \in \Theta_i} P(\tilde{\theta}_i) U_i(\hat{\sigma}_i, \sigma_{-i}^*|\tilde{\theta}_i)\\
&= \sum_{\tilde{\theta}_i \in \Theta_i \setminus \{\theta_i\}} P(\tilde{\theta}_i) U_i(\hat{\sigma}_i, \sigma_{-i}^*|\tilde{\theta}_i) + P(\theta_i) U_i(\hat{\sigma}_i, \sigma_{-i}^*|\theta_i)\\
&= \sum_{\tilde{\theta}_i \in \Theta_i \setminus \{\theta_i\}} P(\tilde{\theta}_i) U_i(\sigma_i^*, \sigma_{-i}^*|\tilde{\theta}_i) + P(\theta_i) U_i(a_i, \sigma_{-i}^*|\theta_i)\\
&> \sum_{\tilde{\theta}_i \in \Theta_i \setminus \{\theta_i\}} P(\tilde{\theta}_i) U_i(\sigma_i^*, \sigma_{-i}^*|\tilde{\theta}_i) + P(\theta_i) U_i(\sigma_i^*, \sigma_{-i}^*|\theta_i) && \text{(from Eqn. 13.8)}\\
&= U_i(\sigma_i^*, \sigma_{-i}^*).
\end{aligned}$$

Which is a contradiction to $(\sigma_i^*, \sigma_{-i}^*)$ being a Nash equilibrium. Thus, our assumption was incorrect and we have proved the theorem. ■

13.4 Existence of Bayesian Equilibrium

Theorem 13.4 *Every finite Bayesian game has a Bayesian equilibrium.*

Proof: *Idea:* Transform the Bayesian game into a complete information normal form game treating each type a player. The transformed game is represented by $\langle \bar{N}, (A_{\theta_i})_{\theta_i \in \bar{N}}, (U_{\theta_i})_{\theta_i \in \bar{N}} \rangle$, where

$$\begin{aligned} \bar{N} &= \cup_{i \in N} \Theta_i = \{\theta_1^1, \theta_1^2, \ldots, \theta_1^{|\Theta_1|}, \theta_2^1, \theta_2^2, \ldots, \theta_2^{|\Theta_2|}, \theta_n^1, \theta_n^2, \ldots, \theta_n^{|\Theta_n|}\} \qquad \text{(finite by assumption)} \\ A_{\theta_i} &= A_i, \ \forall \theta_i \in \Theta_i, \forall i \in N \\ U_{\theta_i}(a_{\theta_i}, a_{-\theta_i}) &= \sum_{\theta_{-i} \in \Theta_{-i}} P(\theta_{-i}|\theta_i) U_i(a_i(\theta_i), a_{-i}(\theta_{-i}), \theta_i, \theta_{-i}) \end{aligned}$$

Note: A mixed strategy of player θ_i, σ_{θ_i} is a probability distribution over ΔA_i, which is a mixed strategy of player i at type θ_i, $\sigma_i(\theta_i)$ in the original Bayesian game. Similarly, we can show that a MSNE in the transformed game is a Bayesian equilibrium in the original game. Since, by Nash theorem, MSNE exists in the transformed game (which is finite), Bayesian equilibrium exists in the original game. ■