CS698W: Game Theory and Collective Choice **Jul-Nov 2017**

Lecture 12: August 29, 2017

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Disclaimer: *These notes aggregate content from several texts and have not been subjected to the usual scrutiny deserved by formal publications. If you find errors, please bring to the notice of the Instructor at swaprava@cse.iitk.ac.in.*

12.1 Introduction

In the complete information games that we have seen so far, the only uncertainties were over the probabilistic choice of actions for different players (e.g. for mixed strategies or behavioral strategies). Even in that case, the payoffs for different paths of a game were known deterministically to every player (was actually a common knowledge). In incomplete information games, as we shall see, that is not the case.

Definition 12.1 (Incomplete Information Games) *An* incomplete information game *is a game where players do not deterministically know which game they are playing. All players may have private information or* types*.*

Consider the following example.

Example 12.2 (A Soccer Game) *Suppose that there is a match scheduled between two competing clubs. The actions available to both the teams is the kind of game they decide to play. Let their choices be either to play attacking / aggressive or to play defensive. Let both these actions be denoted by A and D respectively.*

Now, based on external random factors like weather at the day of game, unexpected player injuries and such, each team may have a hidden agenda. Say, when the teams are in a favourable position, they aim to ***Win (W)*** *while in case of unfavourable position they aim to settle for a* ***Draw (D)****.*

Note: *The agenda of each team is their private information and is unknown to the other team. It is also beyond the control of the players (in this case, the teams). This private information which is a random realization is known as the* type *of the player. The players can privately observe them but cannot choose them.*

Consider the following utilities for different type profiles*.*

WW		
	A	*D*
A	*1,1*	*2,0*
D	*0,2*	*0,0*

WD		
	A	*D*
A	*2,1*	*3,0*
D	*1,1*	*1,0*

DW		
	A	*D*
A	*1,2*	*1,1*
D	*0,3*	*0,1*

DD		
	A	*D*
A	*0,0*	*1,0*
D	*0,1*	*-1,-1*

In this example, there are 4 possible type profiles of the players: WW is the type profile when both teams intend to win and the payoffs from the actions chosen by the players are depicted in the first matrix above. Similarly, the games corresponding to the other type profiles are depicted in the other matrices. Thus, there are 4 different payoff matrices, one corresponding to each type profile. However, even though both players know all the 4 matrices, they do not know which matrix the game they are in. Thus, they do not deterministically know which game they are playing.

We assume that the players and actions available to every player remain the same in different type profiles of the game – the utility is the only thing that changes across type profiles. We consider a special class of the incomplete information games, known as the Bayesian game, where type profiles are chosen from a common prior distribution P.

12.2 Bayesian Games

Formally, a Bayesian game is represented by a 5-tuple.

$$\langle N, (A_i)_{i \in N}, (\Theta_i)_{i \in N}, P, (\Gamma_\theta)_{\theta \in \times_{i \in N} \Theta_i = \Theta} \rangle$$

Where,

N	$= \{1, \ldots, n\}$, set of players.
A_i	: action set of player i.[1]
Θ_i	: set of types for player i; e.g. Win / Draw in Example 12.2.
P	: common prior distribution over $\Theta = \times_{i \in N} \Theta_i$ with the restriction that $P(\theta_i) > 0 \ \forall \theta_i \in \Theta_i, \ \forall i \in N$.[2]
Γ_θ	$:= \langle N, (A_i)_{i \in N}, (u_i(\theta))_{i \in N} \rangle$, a normal form game for the type profile θ.
u_i	$: \Theta \times A \mapsto \mathbb{R}$, where $A = \times_{i \in N} A_i$, normal form game utility function for each type profile.

12.2.1 Stages in Bayesian Games

- $\theta = (\theta_i, \theta_{-i})$ is chosen according to P.
- Each player observes his / her own θ_i.
- They pick action $a_i \in A_i$.
- Player i's payoff $u_i((a_i, a_{-i}); (\theta_i, \theta_{-i}))$ is realized.

12.2.2 Strategies and Utilities

Strategy in a Bayesian game is a plan to map a state/type into action.

Pure Strategy	$s_i : \Theta_i \mapsto A_i$
Mixed Strategy	$\sigma_i : \Theta_i \mapsto \Delta(A_i)$

Definition 12.3 (Ex-ante Utility) *Ex-ante utility of player i is the expected utility before she observes her own type, i.e.,*

$$U_i(\sigma_i, \sigma_{-i}) = \sum_{\theta \in \Theta} P(\theta) \cdot U_i((\sigma_i(\theta_i), \sigma_{-i}(\theta_{-i})), \theta).$$

Where,

$$U_i((\sigma_i(\theta_i), \sigma_{-i}(\theta_{-i})), \theta) = \sum_{(a_1, \ldots, a_n) \in A} \left(\prod_{j \in N} \sigma_j(\theta_j, a_j) \right) u_i((a_i, a_{-i}); (\theta_i, \theta_{-i})). \tag{12.1}$$

In Bayesian games, we assume that once player i observes her type, she establishes a belief according to Bayes rule (hence called a Bayesian game) on P as follows.

$$P(\theta_{-i} | \theta_i) = \frac{P(\theta_i, \theta_{-i})}{\sum_{\tilde{\theta}_{-i} \in \theta_{-i}} P(\theta_i, \tilde{\theta}_{-i}}).$$

Note: This is where the positive marginal is crucial.

Definition 12.4 (Ex-interim Utility) *Ex-interim utility of player i is the expected utility after she observes her own type, i.e.,*

$$U_i(\sigma|\theta_i) = \sum_{\theta_{-i}\in\Theta_{-i}} P(\theta_{-i}|\theta_i) \cdot U_i(\sigma(\theta), \theta).$$

Where $U_i(\sigma(\theta), \theta)$ is as defined in Equation 12.1.

Ex-interim utility is most commoly used for analysis of incomplete information games. The above two utilities are related as follows.

$$U_i(\sigma) = \sum_{\theta_i\in\Theta_i} P(\theta_i) U_i(\sigma|\theta_i).$$

Example 12.5 (Two player bargaining game) *Consider a game between two players where player 1 is the seller and player 2 is a buyer. Player 1's type is the minimum price at which he is willing to sell a commodity. Player 2's type is the maximum amount that the player is willing to pay. For simplicity, let the types of both players be integers. Hence,*

- $N = \{1, 2\}$.
- $\Theta_1 = \Theta_2 = \{1, 2, \ldots, 100\}$.

Assume that both players bid a number in $\{1, 2, \ldots, 100\}$. If the bid of the seller is less than or equal to the bid of the buyer, the sale happens. Else, there is no trade. Hence the action sets of the players are, $A_1 = A_2 = \{1, 2, \ldots, 100\}$. Assume the beliefs of players and the payoffs are

- $P(\theta_2|\theta_1) = \frac{1}{100} \quad \forall\theta_2 \in \Theta_2, \forall\theta_1 \in \Theta_1$
- $P(\theta_1|\theta_2) = \frac{1}{100} \quad \forall\theta_1 \in \Theta_1, \forall\theta_2 \in \Theta_2$
- $u_1(a_1, a_2; \theta_1, \theta_2) = \begin{cases} \frac{a_1+a_2}{2} - \theta_1 & \text{if } a_2 \geq a_1 \\ 0 & \text{otherwise} \end{cases}$
- $u_2(a_1, a_2; \theta_1, \theta_2) = \begin{cases} \theta_2 - \frac{a_1+a_2}{2} & \text{if } a_2 \geq a_1 \\ 0 & \text{otherwise} \end{cases}$

The beliefs $P(\theta_2|\theta_1)$ and $P(\theta_1|\theta_2)$ are consistent with the prior $P(\theta) = \frac{1}{10000}$, $\forall\theta \in \Theta$, where $\Theta = \Theta_1 \times \Theta_2$.

Example 12.6 (Sealed-Bid Auction) *In this game, we have an auctioneer (who is not a player) willing to sell a item via an auction and two buyers (these are the competing players) who place sealed bids (secret to each other) on the item. The player who has bid more is given the item for the amount he/she has bid for.*

The type of each player is the value they attach to the item. Assume that the values belong to $[0, 1]$. Hence $\Theta_1 = \Theta_2 = [0, 1]$. Bids also belong to $[0, 1]$, hence this is same as the players' action set. The allocation functions are

$$o_1(b_1, b_2) = \begin{cases} 1 & \text{if } b_1 \geq b_2 \\ 0 & \text{otherwise} \end{cases}$$

$$o_2(b_1, b_2) = \begin{cases} 1 & \text{if } b_2 > b_1 \\ 0 & \text{otherwise} \end{cases}$$

The probability distribution over this continuous range is given by

$$f_1(\theta_2|\theta_1) = 1, \quad \theta_2 \in [0,1]$$
$$f_2(\theta_1|\theta_2) = 1, \quad \theta_1 \in [0,1]$$

which are consistent with the joint distribution, $f(\theta_1, \theta_2) = 1 \quad (\theta_1, \theta_2) \in [0,1]^2$.

The utility representation for the game is

$$u_i(b_1, b_2; \theta_1, \theta_2) = o_i(b_1, b_2) \cdot (\theta_i - b_i), \; i = 1, 2.$$

Note: *In this utility representation we assume that the valuation and bid currency are in the same metric (basically, we assume that the valuation/satisfaction of buyer for a item can be equated with money). Such utility representations are called as quasi-linear utility representation.*