CS698W: Game Theory and Collective Choice **Jul-Nov 2017**

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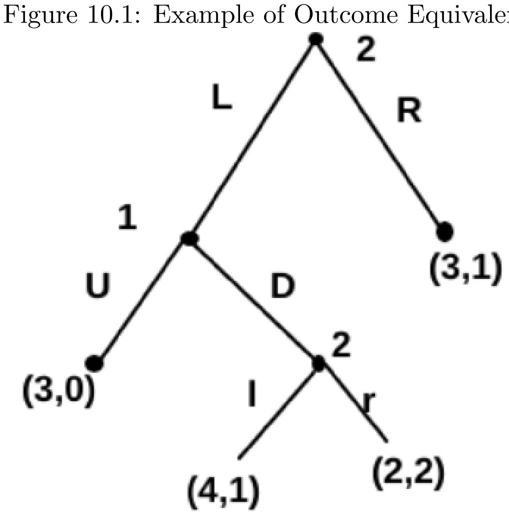
10.1 Outcome Equivalence of Behavioral and Mixed Strategy

In the context of IIEFGs *behavioral strategy* of a player is a probability distribution over the actions at every information set of the player. Thus the player is randomizing at each node rather than randomizing over the pure strategies of an IIEFG (the complete contingency plan). The latter is a mixed strategy of the player.

Definition 10.1 (Outcome Equivalence) *A behavioral strategy b_i and a mixed strategy σ_i are* outcome equivalent *if for all σ_{-i}, the probability distribution induced over the terminal vertices are the same for (b_i, σ_{-i}) and (σ_i, σ_{-i}).*

Figure 10.1: Example of Outcome Equivalence



In the above game, consider the following mixed strategy of player 2:

$$\sigma_2(Ll) = \sigma_2(Lr) = \frac{1}{3},\ \sigma_2(Rl) = \frac{1}{12},\ \sigma_2(Rr) = \frac{1}{4}.$$

Also, consider the following behavioral strategy of player 2:

$$b_2(L) = \frac{2}{3},\ b_2(R) = \frac{1}{3},\ b_2(l) = \frac{1}{2} = b_2(r).$$

Suppose player 1 plays U with probability P_U and D with probability P_D. Now for the probability of reaching terminal history LU, according to the mixed strategy is equal to the probability of reaching the same terminal history w.r.t. the behavioral strategy, i.e.,

$$(\sigma_2(Ll) + \sigma_2(Lr))P_U = \frac{2}{3}P_U = b_2(L) \cdot P_U.$$

Similar conclusions hold for the terminal histories LDl, LDr, and R.

$$\begin{aligned} \text{For LDl:} \quad & \sigma_2(Ll)P_D = \frac{1}{3}P_D = b_2(L) \cdot P_D \cdot b_2(l) \\ \text{For LDr:} \quad & \sigma_2(Lr)P_D = \frac{1}{3}P_D = b_2(L) \cdot P_D \cdot b_2(r) \\ \text{For R:} \quad & (\sigma_2(Rl) + \sigma_2(Rr))P_U = \frac{1}{3} \cdot P_U = b_2(R) \cdot P_U \end{aligned}$$

Thus we see that all the terminal nodes have same probability of occurence under the mixed strategies and behavioral strategies. Thus, the abovementioned mixed strategy and behavioral strategy for player 2 are outcome equivalent. The following result shows that this equivalence always holds for games of perfect recall.

Theorem 10.2 (Kuhn 1953) *In IIEFGs with perfect recall every mixed strategy is outcome equivalent to behavioral strategies.*

The proof of the theorem is constructive and follows the principle used in the example above. From now on, we will consider games only with perfect recall, and all strategies will be referred to as behavioral strategies.

10.2 Equilibrium Notion and Importance of Belief

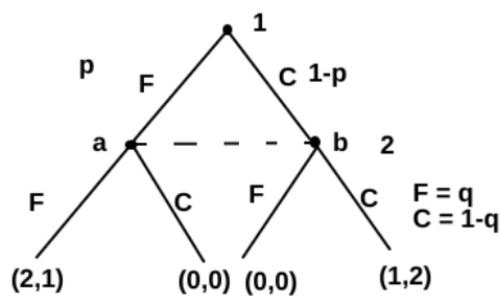


Figure 10.2: Football-Cricket Game for 2 players

In the example of Football - Cricket game (Figure 10.2)

- if player 2 believes that, $p_a > \frac{2}{3}$, then it is better for him to play F.
- if player 2 believes that, $p_a < \frac{2}{3}$, then it is better for him to play C.
- if player 2 believes that, $p_a = \frac{2}{3}$, then he can mix F and C in any way.

Thus, the players' belief is important in deciding his/her strategy. In the equilibrium notion of IIEFGs, this idea is made explicit.

10.2.1 Belief

Let the *information sets* of player i be $I_i = \{I_i^1, I_i^2, ..., I_i^{k(i)}\}$. In an IIEFG, the belief of player i is a map $\mu_i^j : I_i^j \to [0, 1]$, such that,

$$\sum_{x \in I_i^j} \mu_i^j(x) = 1.$$

10.2.2 Bayesian Belief of Player i

A belief $\mu_i := (\mu_i^j, j = 1, \dots, k(i))$ of player i is Bayesian with respect to the behavioral strategy σ, if it is derived from the strategy profile σ using Bayes rule, i.e.,

$$\mu_i^j(x) = \frac{P_\sigma(x)}{\sum_{y \in I_i^j} P_\sigma(y)}, \forall x \in I_i^j,\ \forall j = 1, \dots, k(i).$$

10.3 Sequential Rationality

A strategy σ_i of player i at an information set I_i^j is sequentially rational given σ_{-i} and beliefs μ_i if $\forall \sigma_i'$

$$\sum_{x \epsilon I_i^j} \mu_i(x) u_i(\sigma_i, \sigma_{-i}|x) \geq \sum_{x \epsilon I_i^j} \mu_i(x) u_i(\sigma_i', \sigma_{-i}|x).$$

10.4 Football Cricket Game example for Sequential Rationality

In figure 10.2 we see that the game is played between two players 1 and 2. Players 1 and 2 picks action F with probabilities p and q respectively. Consider a belief (we suppress the superscript since each player has only one information set)

$$\mu_2(a) = p = 0.5$$
$$\mu_2(b) = 1 - p = 0.5$$

Then, player 2 can compute,

$$\sum_{x \epsilon I_2^1} \mu_2(x) U_2(\sigma_1, \sigma_2|x) = 0.5[q.1 + (1 - q).0] + 0.5[2 \times (1 - q)] = 0.5[2 - q]$$

Thus, to maximize his utility given his belief about the moves of player 1, it will be sequentially rational for player 2 to pick $q = 0$, i.e., play football with zero probability.

So, $\sigma = ((0.5, 0.5), (0, 1))$ is sequentially rational for player 2. But is it sequentially rational for player 1?

10.5 Perfect Bayesian Equilibrium

An assessment (σ, μ) is a *perfect Bayesian equilibrium* (PBE) if *for every player* i

1. μ_i is Bayesian with respect to σ,
2. σ_i is sequentially rational given σ_{-i} and μ_i at every information set of i.

From the last example, we can find out that σ = ((0.5,0.5), (0,1)) is not sequentially rational for player 1. Hence it is not a PBE. However, ((2/3,1/3), (1/3,2/3)) is.

We provide a result that shows that PBE is a *refinement* of MSNE.

Theorem 10.3 *Every Perfect Bayesian Equilibrium (PBE) is a Mixed Strategy Nash Equilibrium (MSNE).*

Proof: Exercise. ■