CS698W: Game Theory and Collective Choice	Jul-Nov 2017
Lecture 9: August 22, 2017	
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9.1 Imperfect-information extensive-form games

A PIEFG is not able to represent the simultaneous move games like neighboring kingdoms dilemma. Hence we needed to move to a more general representation, namely imperfect information EFG.

Definition 9.1 (Imperfect Information Extensive Form Game) *An imperfect-information extensive form game is a tuple*

$$\langle N, A, \mathcal{H}, \mathcal{X}, P, (u_i)_{i\in N}, (I_i)_{i\in N}\rangle$$

Where $\langle N, A, \mathcal{H}, \mathcal{X}, P, (u_i)_{i\in N}\rangle$ is a PIEFG and for every $i \in N$, $I_i := (I_i^1, I_i^2, \dots, I_i^{k(i)})$ is a partition of $\{h \in \mathcal{H} \setminus Z : P(h) = i\}$ with the property that if $h, h' \in I_i^j$, then $\mathcal{X}(h) = \mathcal{X}(h')$. The sets in the partition I_i are called information sets *of player i, and in a specific information set, the actions available to player i are same.*

Set I_i for every player i, is a collection of Information sets $I_i^j, j = 1, \dots, k(i)$. Information sets are collection of histories where the player at that history is uncertain about which history has been reached.

Since by Definition 9.1, the actions at an information set are identical, we can define $\mathcal{X}$ over Information sets I_i^js, rather than defining them over histories h, h'. Therefore

$$\mathcal{X}(h) = \mathcal{X}(h') = \mathcal{X}(I_i^j).$$

Definition 9.2 (Strategy Set) *Strategy set of player $i, i \in N$ is defined as the Cartesian product of the actions available to player i at his information sets, i.e.,*

$$S_i = \bigtimes_{\tilde{I}\in I_i} \mathcal{X}(\tilde{I}).$$

The intuition of the information set comes from the fact that a player can be uncertain about the true state of a game when he is deciding his action. Consider the game of neighboring kingdoms dilemma (Fig. 9.1). In IIEFG representation, Player 2 does not know at which node (a or b), he presently is. Thus his action here is independent of it.

9.1.1 Representational equivalence

To draw a connection between different representations of games, we need the equilibrium concepts to follow in newer representations too. However, in IIEFG there could be strategies which are different than a mixed strategy.

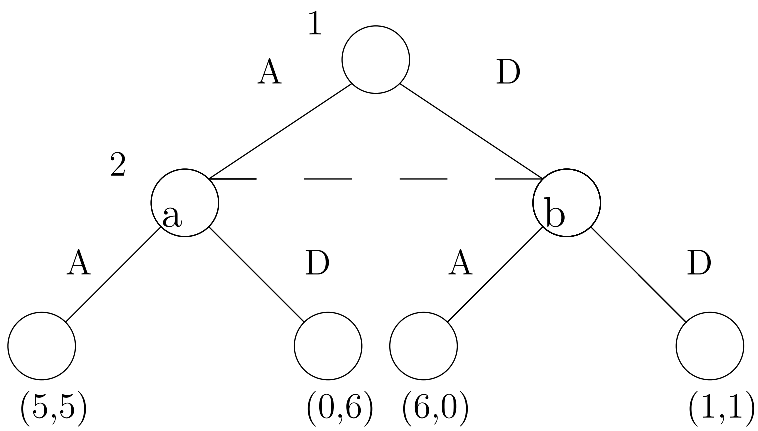

Figure 9.1: Neighboring Kingdom's Dilemma

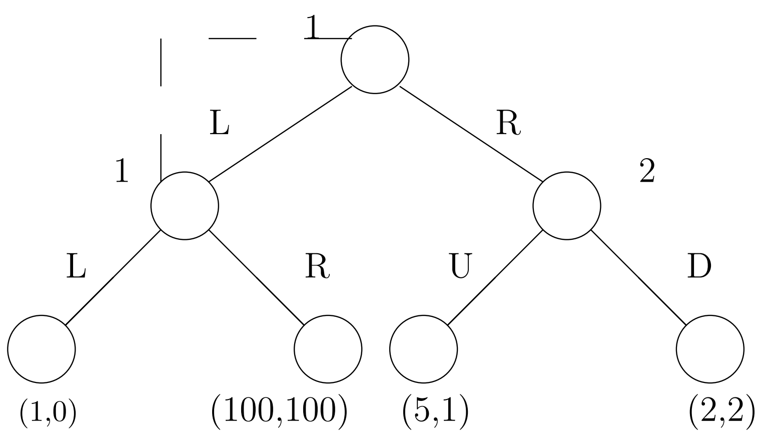

Figure 9.2: IIEFG representation of a game with a forgetful player

Definition 9.3 (Behavioral Strategy) *In a IIEFG, a behavioral strategy assigns a probability distribution over the set of possible actions at each information set.*

This is quite different from the mixed strategies. In a mixed strategy, a player assigns probabilities over the pure strategies (which is a complete contingency plan). But in behavioral strategy, the player independently randomizes over his actions at his every information set.

At this point, it may appear that a transformation between these two strategies is always possible. But we show that these two notions are actually incomparable.

Example 9.1.1 *Consider the IIEFG shown in Figure 9.2. The discontinuous line between the nodes where player 1 plays shows that these two nodes are in the same information set of player 1. Thus his strategy set is* $\{L, R\}$*. Player 2 has two strategies* U *and* D*.*

Mixed strategy *From the NFG representation of this game, we can conclude that for player 1 R strictly dominates L, and for player 2 D weakly dominates U. Thus game have* ***weakly dominant strategy equilibrium*** (R, D)*, which is also the unique MSNE given by* $((0, 1), (0, 1))$*.*

Behavioral strategy *In behavioral strategy, players randomize over actions at each information set. From IIEFG representation we can see that player 2's rationality must dictate him to play D, as it is always gives him more payoff. So, with the probability distribution for player 2 being $(0 : U, 1 : D)$, let the probability distribution for player 1 be $(p : L, (1-p) : R)$. The expected payoff for player 1 is*

$$p^2 \times 1 + p(1-p) \times 100 + (1-p) \times 2,$$

which is maximum at $p = \frac{98}{198}$. Hence, a behavioral strategy equilibrium for this game is $((\frac{98}{198}, \frac{100}{198}), (0, 1))$.

MSNE and BSE are different in this game because player 1 forgets what he played at level 1. This kind of games is known as IIEFGs with imperfect recall. Next, we consider games with perfect recall, where these two strategies are equivalent.

9.2 Games with Perfect Recall

Let us consider type of games where at every opportunity of a player to act, he remembers exactly what he did at his every previous turn to play. Such games are called *games with perfect recall.*

Definition 9.4 *Player i has a perfect recall in an IIEFG, if for any two histories $h, h' \in I_i^j$ where $h = (v_0, a_0, v_1, a_1, ..., v_{m-1}, a_{m-1}, v_m)$ and $h' = (v'_0, a'_0, v'_1, a'_1, ..., v'_{n-1}, a'_{n-1}, v'_n)$ where $v_i, v'_i s$ are the nodes of the IIEFG and $a_i, a'_i s$ are the actions at corresponding nodes, the following holds.*

1. $m = n$
2. *for all j s.t. $0 \leq j \leq (m-1)$, v_j, v'_j must be in same information set of player i.*
3. *for all j s.t. $0 \leq j \leq (m-1)$, **if** $P(h_j) = i$ **then** $a_j = a'_j$, where h_j is the truncated history h at level j from the root.*

*A game has **perfect recall** if every player has perfect recall.*

References

[CW87] M. Jackson, K. Leyton- Brown and Y. Shoham, Game Theory Lecture 4-8 "Imperfect Information Extensive Form: Definition, Strategies"