CS698W: Game Theory and Collective Choice **Jul-Nov 2017**

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Disclaimer: *These notes aggregate content from several texts and have not been subjected to the usual scrutiny deserved by formal publications. If you find errors, please bring to the notice of the Instructor at swaprava@cse.iitk.ac.in.*

8.1 Recap

Extensive form games are used to represent the sequential games. We will only consider perfect information extensive form games (PIEFG) with a *finite* number of stages. It can always be transformed into normal form game and find the Nash equilibrium. However, as we have seen, this representation is often redundant and the NE prediction may lead to non-credible threats. Therefore, we need a new equilibrium notion, which is a *refinement* of Nash equilibrium. A refinement of a Nash equilibrium refers to an equilibrium which is NE but the converse may not hold. We have the following intuitive result, which will be obvious when we define the refined equilibrium.

Theorem 8.1 *Every finite PIEFG has a PSNE.*

Here, by finite PIEFG, we mean that the number of players, actions, and stages of the PIEFG are finite.

Intuition: At every stage of the game, a player has *perfect information* about the action taken by the former player. Hence there is no reason for *randomizing* over the actions. There is always a pure action that is *weakly* superior than the other actions. This will be obvious after the discussion on subgame perfection.

Examples of PIEFG: Chess, Tic-Tac-Toe, Bargaining.

Zermelo (1913) showed using an argument similar to PIEFG that if both the players in chess are infinitely rational and intelligent, chess must be a very boring game.

In the given perfect information extensive normal game $((BH), (CE))$ is indeed a PSNE but it is not very intuitive. Player 1 at his second choice node plays H instead of G whereas he will get more utility had he played G. This suspicious behavior of Player 1 is termed as a *threat* to Player 2 forcing him to play E (as the unilateral deviation of Player 2 to F will yield an utility of 0 to him). Hence it continues to be a Nash equilibrium. But this threat of Player 1 is really not *credible*. If Player 2 ever plays F instead of E and the game reaches the stage where Player 1 has options between G and H, will Player 1 still choose H over G (at his own loss)?

Therefore, the Nash equilibrium $((BH), (CE))$ seems ambiguous in the context of PIEFGs. Hence, there is a need to refine the concept of equilibrium through defining the notion of a subgame to avoid such equilibrium having non-credible threats.

Definition 8.2 (Subgame) *The extensive form game represented by the subtree at a node is called the subgame at that node.*

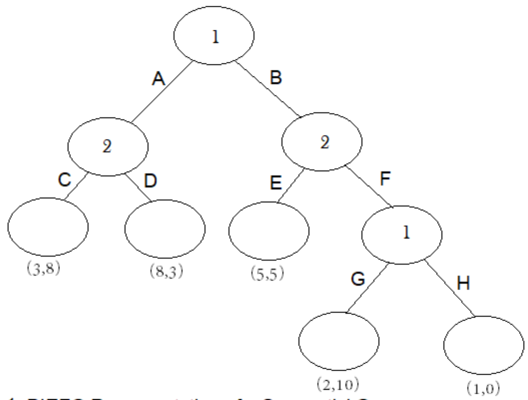


Figure 8.1: PIEFG representation of a sequential game

Definition 8.3 (Subgame Perfect Nash Equilibrium) *The* subgame perfect Nash equilibrium *(SPNE) of a game G are all strategy profiles $s \in S := \bigtimes_{i \in N} S_i$ such that for any subgame G' of G the restriction of s to G' is a Nash equilibrium of G'.*

We define the restriction of a strategy profile s to a subgame G (denoted by $s|G'$) as the truncation of the actions of s to the actions that are relevant in G'. For example, in Figure 8.1, let $s = ((BH), (CE))$ and G' be the subtree rooted at the second decision node of Player 1 (at history BF), then $s|G' = (H, \varnothing)$. Clearly, $s|G'$ is not a NE at G', hence $((BH), (CE))$ is not an SPNE.

8.2 Computing SPNE: Backward Induction Algorithm

1. Start at the leaf having maximum depth.
2. For the player in the parent node find the action which maximizes the utility for that player.
3. Retain that (player,action) and delete all the edges at that level, translate the utilities to the parent node.
4. Go up one level and repeat (1).
5. Stop if root is reached.

The algorithm is implemented as a single depth first traversal of the game tree. It identifies the strategy which has higher pay-off for the player in the bottom-most subgame tree and removes the other counter actions and updates the pay-off of the parent node and repeats the complete process again. The algorithm terminates when the action with higher pay-off among all the possible actions for the player at root is found out. In this manner, the algorithm gives SPNE.

The algorithm yields much better prediction than the Nash equilibria since the treat in SPNE is credible. Also the algorithm always gives an SPNE. Therefore, SPNE is guaranteed to exist in a PIEFG. Though, there could be multiple SPNEs.

However, the complexity of BI algorithm sometimes prohibitive – since it needs to parse all possible paths in the game tree. For games like chess, the extensive form representation has $\sim 10^{150}$ nodes and it is not feasible to apply backward induction. Game softwares uses heuristic pruning for computer players and does not consider the parts of the game tree which can never be the candidates for equilibrium.

Exercise: check if $((AG), (CF))$ is an SPNE of the previous game.

Note: SPNE is always a PSNE but the reverse is not true.

8.3 Limitations of SPNE

Centipede Game: In this game two players makes alternate decisions, at each turn choosing between going "down" and ending the game or going "across" and continuing it except at the last node where going "across" also ends the game. The payoffs are constructed in such a way that the player achieves higher payoffs by choosing "down". Consider the last choice, at that point the best choice for the player is to go down. Going down is also the best choice for the other player in the previous choice point. By induction the same argument holds for all choice points.

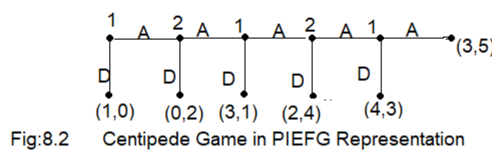


Figure 8.2: Centipede game

When this game was experimented on people it was found that people played "across" moves also until close to end of the game. This is a practical criticism of the concept of SPNE.

On the theory side, suppose that you are the second player in the game, and in the first step of the game the first player plays across. The SPNE tells that you should go down, but the same analysis suggests that you would not have reached this choice point in the first place. Hence the idea of SPNE is self-contradictory at certain decision nodes, even though it guarantees NE at every subgame.

8.4 Imperfect Information Extensive Form Games

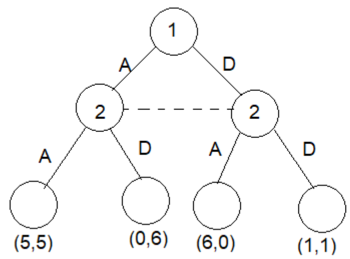
The PIEFG is not able to represent the simultaneous move games like neighboring kingdom's dilemma. Hence there is a need to move for a more general representation.

Definition 8.4 (Imperfect Information Extensive Form Game) *An imperfect-information extensive form game is a tuple*

$$\langle N, A, \mathcal{H}, \mathcal{X}, P, (u_i)_{i\in N}, (I_i)_{i\in N}\rangle$$

Where $\langle N, A, \mathcal{H}, \mathcal{X}, P, (u_i)_{i\in N}\rangle$ is a PIEFG and for every $i \in N$, $I_i := (I_i^1, I_i^2, \ldots, I_i^{k(i)})$ is a partition of $\{h \in \mathcal{H} \setminus Z : P(h) = i\}$ with the property that if $h, h' \in I_i^j$, then $\mathcal{X}(h) = \mathcal{X}(h')$. The sets in the partition I_i are called information sets *of player i, and in a specific information set, the actions available to player i are same.*

In the game shown in Figure 8.3, both players 1 and 2 have one information set each, and $I_1^1 = \{\varnothing\}$, $I_2^1 = \{(A), (D)\}$.



Imperfect-information extensive form game Representation of Neighboring Kingdom Dilemma

1 \ 2	A	D
A	5,5	0,6
D	6,0	1,1

Normal Form Representation

Figure 8.3: An example EFG with imperfect information

An information set is always non-empty but it can be a singleton as PIEFG is also a IIEFG with singleton information sets. IIEFG is a richer representation than PIEFG as an NFG can also be converted into IIEFG.