CS698W: Game Theory and Collective Choice **Jul-Nov 2017**

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Disclaimer: *These notes aggregate content from several texts and have not been subjected to the usual scrutiny deserved by formal publications. If you find errors, please bring to the notice of the Instructor at swaprava@cse.iitk.ac.in.*

7.1 Correlated Equilibrium

Motivation In the previous lectures, we have seen games with different types of equilibria and finally arrived at the mixed strategy Nash equilibrium (MSNE) which was the weakest and most general. The most special property of MSNE is that it always exists for any finite game and can be found by solving a finite number of (potentially non-linear) equations. However, calculating an MSNE is computationally difficult. In this lecture, we look at another equilibrium notion called *correlated equilibrium* (CE) which is weaker than MSNE.

In a Nash equilibrium, each player chooses his strategy independent of the other player, which may not always lead to the best outcome. However if the players trust a third-party agent, who randomizes over the strategy profiles and suggests the individual strategies to the concerned players, the outcomes can be significantly better. Such a strategy is called correlated strategy. Note that a correlated strategy is *not* a strategy of the players, rather it is a strategy of the third-party agent.

Example 7.1.1 *Consider a busy crossing of two roads. If the traffic of both roads move at the same time, it is a chaos, leading to potential accidents. If both stops, it is useless. The only good outcome is when traffic on one road stops and the other moves. The traffic light accomplishes this objective by periodically asking one road to stop and the other road to move. The traffic light serves the purpose of the (automated) third party agent and the players are the traffic of each road.*

7.2 Definition and Examples

Definition 7.1 *A* correlated strategy *is a mapping* $\pi : S \mapsto [0,1]$ *such that* $\sum_{s \in S} \pi(s) = 1$ *where* $S = S_1 \times S_2 \times \cdots \times S_n$ *and* S_i *represents the strategy set of player* i*.*

Hence, a correlated strategy π is a joint probability distribution over the strategy profiles.

A correlated strategy is a correlated equilibrium if it becomes self-enforcing, i.e., no player 'gains' by deviating from the suggested strategy.

Note: Here the suggested strategy π is a common knowledge.

Definition 7.2 *A* correlated equilibrium *(CE) is a correlated strategy* π *such that* $\forall s_i \in S_i$ *and* $\forall i \in N$,

$$\sum_{s_{-i} \in S_{-i}} \pi(s_i, s_{-i}) u_i(s_i, s_{-i}) \geq \sum_{s_{-i} \in S_{-i}} \pi(s_i, s_{-i}) u_i(s_i', s_{-i}) \qquad \forall s_i' \in S_i. \tag{7.1}$$

This means that the player i does not gain any advantage in the expected utility if he deviates from the suggested correlated strategy π, assuming all other players follow the suggested strategy.

Another way to interpret a correlated equilibrium is that π is a single randomization device (a dice e.g.) which gives a random outcome which is a strategy profile, and a specific player only observes the strategy corresponding to her. Given that observation, she computes her expected utility, and if that does not improve if she picks another strategy (and it happens for every player) that randomization device is a correlated equilibrium.

The following examples help us understand the definitions better.

7.2.1 Game Selection Problem

In the problem, two friends want to go to watch a game together, however Player 1 like Cricket more and Player 2 likes Football. The Utility Function is represented in the form:

1 \2	C	F
C	2,1	0,0
F	0,0	1,2

In MSNE, we saw that the expected utility of each player was $\frac{2}{3}$. However if the correlated strategy is such that $\pi(C,C) = \frac{1}{2} = \pi(F,F)$. If we assume that Player 1 is suggested to choose F, then the expected utility from following the suggestion is given as

$$\sum_{s_{-1}\in S_{-1}} \pi(s_{-1}|F)u_1(F,s_{-1}) = \frac{1}{\pi(F)}[\pi(F,C)u_1(F,C) + \pi(F,F)u_1(F,F)] = \frac{1}{\frac{1}{2}}\left[0 + \frac{1}{2}1\right] = 1 \tag{7.2}$$

where $\pi(F)$ is the probability that F is suggested to Player 1, $\pi(s_{-1}|F)$ is the probability that s_{-1} is strategy of other players when 1 is suggested F and $\pi(F,F)$ is probability that (F,F) is the strategy profile. If Player 1 deviates from the strategy, then his expected utility is

$$\sum_{s_{-1}\in S_{-1}} \pi(s_{-1}|F)u_1(C,s_{-1}) = \frac{1}{\pi(F)}\left[\pi(C,F)u_1(C,C) + \pi(F,F)u_1(C,F)\right] = \frac{1}{\frac{1}{2}}[0+0] = 0. \tag{7.3}$$

Similarly, if C is suggested to Player 1, his expected utility is 2 when he follows the suggestion and 0 when he does not follow. Similar conclusions hold when we consider Player 2. This proves that the correlated strategy here is a correlated equilibrium.

Note that utility at the expected equilibrium is $\frac{1}{2}(1+2) = \frac{3}{2}$ as compared to $\frac{2}{3}$ in MSNE.

7.2.2 Traffic Accident Problem

In the problem, two cars are at a crossroad and wish to cross it. Their utilities are positive if they cross the road, by they cross together, they will collide. The utilities is represented in the form:

1 \2	Stop	Go
Stop	0,0	1,2
Go	2,1	-10,-10

Consider a correlated strategy π such that $\pi(S,G)=\pi(S,S)=\pi(G,S)=\frac{1}{3}$. If we assume that Player 1 is suggested to choose Stop, then the expected utility from following the suggestion is given as

$$\sum_{s_{-1}\in S_{-1}} \pi(s_{-1}|S)u_1(S,s_{-1}) = \frac{1}{\pi(S)}[\pi(S,S)u_1(S,S)+\pi(S,G)u_1(S,G)] = \frac{1}{\frac{2}{3}}\left[0+\frac{1}{3}1\right] = \frac{1}{2}. \tag{7.4}$$

where $\pi(S)$ is the probability that S is suggested to Player 1, $\pi(s_{-1}|S)$ is the probability that s_{-1} is strategy of other players when 1 is suggested S and $\pi(S,S)$ is probability that (S,S) is the suggested strategy. If Player 1 deviates from the strategy, then his expected utility is

$$\sum_{s_{-1}\in S_{-1}} \pi(s_{-1}|S)u_1(G,s_{-1}) = \frac{1}{\pi(S)}[\pi(S,S)u_1(G,S)+\pi(S,G)u_1(G,G)] = \frac{1}{\frac{2}{3}}\left[\frac{1}{3}2+\frac{1}{3}(-10)\right] = -4. \tag{7.5}$$

If G is suggested to Player 1, the expected utility is 2 on following the suggestion and 0 on deviating from the suggestion. Similarly, we can find that same conclusions hold when we consider Player 2. This proves that the correlated strategy here is a correlated equilibrium.

This game is known as Chicken Game as well.

Note: CE is generally not unique for any game and depends upon the randomization process. In our example itself, $\pi(S,G)=\pi(G,S)=\frac{1}{2}$ is also a CE with expected utility of $\frac{3}{2}$ for each player.

7.2.3 Interpretation

Another way to interpret a CE is that it is a distribution over the strategy profiles such that if a strategy which has a non-zero probability of occurrence for Player i is suggested to i, the player can compute the posterior distribution of the strategies suggested to other players. Player i's expected payoff according to that distribution will be maximized by following the suggestion if other players follow their respective suggestions as well. More formally, let $\bar{s}_i$ be the strategy suggested to Player i, then it is a CE if $\forall i \in N$:

$$\sum_{s_{-i}\in S_{-i}} \pi(s_{-i}|\bar{s}_i)u_i(\bar{s}_i,s_{-i}) \geq \sum_{s_{-i}\in S_{-i}} \pi(s_{-i}|\bar{s}_i)u_i(s'_i,s_{-i}),\ \forall s'_i \in S_i \tag{7.6}$$

$$\Rightarrow \sum_{s_{-i}\in S_{-i}} \pi(\bar{s}_i,s_{-i})u_i(\bar{s}_i,s_{-i}) \geq \sum_{s_{-i}\in S_{-i}} \pi(\bar{s}_i,s_{-i})u_i(s'_i,s_{-i}),\ \forall s'_i \in S_i. \tag{7.7}$$

7.2.4 Computing the Correlated Equilibrium:

To find a CE, we need to solve a set of linear equations with the variables as $\pi(s), s\in S$. By 7.1, we know $\pi(s)$ is a CE if $\forall s_i \in S_i$, and $\forall i \in N$

$$\sum_{s_{-i}\in S_{-i}} \pi(s)u_i(s_i,s_{-i}) \geq \sum_{s_{-i}\in S_{-i}} \pi(s)u_i(s'_i,s_{-i}) \forall s'_i \in S_i. \tag{7.8}$$

The total number of inequalities here are $O(nm^2)$, assuming $|S_i|=m, \forall i\in N$. We also need to ensure that $\pi(s)$ is a valid probability distribution. Therefore

$$\pi(s) \geq 0,\ \forall s \in S \qquad m^n \text{ inequalities} \tag{7.9}$$

$$\sum_{s\in S} \pi(s) = 1 \tag{7.10}$$

The inequalities together represent a feasibility LP which is poly-time solvable. For computing MSNE, the number of support profiles are $O(2^{mn})$, which is exponentially larger than the number of inequalities to find a CE ($O(m^n)$). Therefore computing a CE is a much simpler problem than a MSNE. It can also be shown that every MSNE is a CE. The whole space of equilibrium guarantees can be represented as:

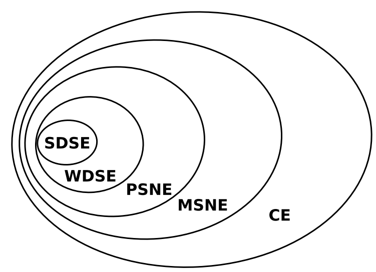

Figure 7.1: Venn diagram of the equilibria of a game

7.3 Extensive Form Games

We now discuss a different form of representing games, that is more appropriate for multistage games. These game representation is called *extensive form games* (EFG). We start with the perfect information EFGs.

7.3.1 Perfect Information Extensive Form Games

Normal form games (NFG) are appropriate when players take their actions simultaneously – their action profile decides the outcome. On the other hand, in **Extensive form games**, players take actions depending upon the sequence of actions, which we will call *history*, taken in the game, and the outcome happens at the end of the sequential actions. While NFGs are easily represented by payoff matrices, EFGs are best represented by a tree-like structure. In a **Perfect Information EFG**, every player knows the history till that time *perfectly.* The game below is an example of PIEFG.

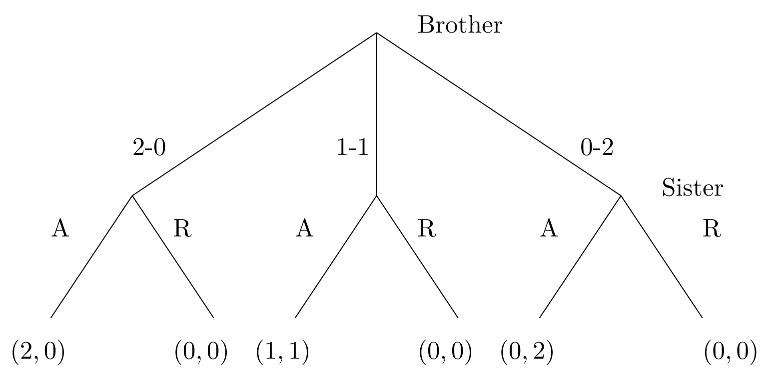

Figure 7.2: Brother-sister chocolate sharing game in extensive form

7.3.2 Chocolate Division Game

Suppose a mother gives his elder son two (indivisible) chocolates to share between him and his younger sister. She also warns that if there is any dispute in the sharing, she will take the chocolates back and nobody will get anything. The brother can propose the following sharing options: (2-0): brother gets two, sister gets nothing, or (1-1): both gets one each, or (0-2): both chocolates to the sister. After the brother proposes the sharing, his sister may "Accept" the division or "Reject" it. This can be represented as shown in Figure 7.2.

7.3.3 Notation

We formally denote a PIEFG by the tuple

$$\langle N, A, \mathcal{H}, \mathcal{X}, P, (u_i)_{i \in N} \rangle$$

where,

N	set of players
A	set of all possible actions (of all players)
$\mathcal{H}$	set of all *sequences of actions* (histories) satisfying
	empty sequence $\varnothing \in \mathcal{H}$
	if $h \in \mathcal{H}$, any initial continuous sub-sequence h' of h belongs to $\mathcal{H}$
	A history $h = (a^{(0)}, a^{(1)}, \ldots, a^{(T-1)})$ is *terminal* if
	$\nexists\, a^{(T)} \in A$ s.t. $(a^{(0)}, a^{(1)}, \ldots, a^{(T-1)}, a^{(T)}) \in H$
$Z \subseteq \mathcal{H}$	set of all *terminal* histories
$\mathcal{X} : H \setminus Z \mapsto 2^A$	action set selection function
$P : H \setminus Z \mapsto N$	player function
$u_i : Z \mapsto \mathbb{R}$	utility function of player i

The *strategy* of a player in an EFG is a sequence of actions at every history where the player plays. Formally

$$S_i = \bigtimes_{\{h \in H : P(h) = i\}} \mathcal{X}(h).$$

In other words, it is a *complete contingency plan* of the player. It enumerates potential actions a player can take at every node where he can play, even though some sequence of actions may never be executed together.

7.3.4 Representing the Chocolate Division Game

With the notation above, we represent the game as follows.

$$
\begin{aligned}
N &= \{B, S\},\ A = \{2-0, 1-1, 0-2, A, R\} \\
\mathcal{H} &= \{\varnothing, (2-0), (1-1), (0-2), (2-0, A), (2-0, R), (1-1, A), (1-1, R), (0-2, A), (0-2, R)\} \\
Z &= \{(2-0, A), (2-0, R), (1-1, A), (1-1, R), (0-2, A), (0-2, R)\} \\
\mathcal{X}(\varnothing) &= \{(2-0), (1-1), (0-2)\} \\
\mathcal{X}(2-0) &= \mathcal{X}(1-1) = \mathcal{X}(0-2) = \{A, R\} \\
P(\varnothing) &= B,\ P(2-0) = P(1-1) = P(0-2) = S \\
u_B(2-0, A) &= 2,\ u_B(1-1, A) = 1,\ u_S(1-1, A) = 1,\ u_S(0-2, A) = 2 \\
u_B(0-2, A) &= u_B(0-2, R) = u_B(1-1, R) = u_B(2-0, R) = 0 \\
u_S(0-2, R) &= u_S(1-1, R) = u_S(2-0, R) = u_S(2-0, A) = 0 \\
S_1 &= \{2-0, 1-1, 0-2\} \\
S_2 &= \{Y, N\} \times \{Y, N\} \times \{Y, N\} = \{YYY, YYN, YNY, YNN, NYY, NYN, NNY, NNN\}
\end{aligned}
$$

7.3.5 Representing PIEFG as NFG

Given S_1 and S_2, we can represent the game as an NFG, which can be written in the form of matrix. This can be generalized for all PIEFG, i.e., each PIEFG can be represented as a NFG. For the given example, we can express the utility function as in the following table:

B \S	YYY	YYN	YNY	YNN	NYY	NYN	NNY	NNN
2-0	(2,0)	(2,0)	(2,0)	(2,0)	(0,0)	(0,0)	(0,0)	(0,0)
1-1	(1,1)	(1,1)	(0,0)	(0,0)	(1,1)	(1,1)	(0,0)	(0,0)
0-2	(0,2)	(0,0)	(0,2)	(0,0)	(0,2)	(0,0)	(0,2)	(0,0)

Observe that there are many PSNEs in the given game, some of which leads to quite nonintuitive solutions. The PSNEs are marked in **Bold**.

B \S	YYY	YYN	YNY	YNN	NYY	NYN	NNY	NNN
2-0	**(2,0)**	**(2,0)**	**(2,0)**	**(2,0)**	(0,0)	(0,0)	**(0,0)**	**(0,0)**
1-1	(1,1)	(1,1)	(0,0)	(0,0)	**(1,1)**	**(1,1)**	(0,0)	(0,0)
0-2	(0,2)	(0,0)	(0,2)	(0,0)	(0,2)	(0,0)	**(0,2)**	(0,0)

Some of the results like {2-0,NNY}, {2-0,NNN} and {0-2} are not practically useful. As the representation is very clumsy and does not provide us with any advantage, the NFG representation is wasteful and the EFG representation is succinct for such cases. The example also forces us to look for some other equilibrium ideas as the Nash Equilibrium does not serve much purpose in this case.