CS698W: Game Theory and Collective Choice **Jul-Nov 2017**

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Disclaimer: *These notes aggregate content from several texts and have not been subjected to the usual scrutiny deserved by formal publications. If you find errors, please bring to the notice of the Instructor at swaprava@cse.iitk.ac.in.*

6.1 Introduction

We have seen discussions on computing Nash equilibrium. In this lecture, we will address a more fundamental question: the existence of a mixed Nash equilibrium. Nash showed that MSNE exists in any finite game. To prove this result, we will use a result from real analysis. First, we discuss some basic definitions of sets that will be used in presenting the result.

6.2 Definitions and Standard Results

- A set $S \subseteq \mathbb{R}^n$ is **convex** if $\forall x, y \in S$ and $\forall \lambda \in [0,1]$, $\lambda x + (1-\lambda)y \in S$.
- A set $S \subseteq \mathbb{R}^n$ is **closed** if it contains all its limit points (points whose every neighborhood contains a point in S – e.g., for the point 1 in the interval $[0,1)$, consider a ball of radius $\epsilon > 0$, arbitrary, clearly, each such ball will contain a point in $[0,1)$).
- A set $S \subseteq \mathbb{R}^n$ is **bounded** if $\exists\ x_0 \in \mathbb{R}^n$ and $R \in (0,\infty)$ such that $\forall x \in S, \| x - x_0 \|_2 < R$.
- A set $S \subseteq \mathbb{R}^n$ is **compact** if it is *closed* and *bounded*.

Now we state the result from real analysis without proof.

Theorem 6.1 (Brouwer's Fixed Point Theorem) *If $S \subseteq \mathbb{R}^n$ is convex and compact and $T : S \mapsto S$ is continuous, then T has a fixed point, i.e., $\exists$ a point $x^* \in S$ s.t. $T(x^*) = x^*$.*

6.3 Existence of MSNE

Finite game: A game in which the number of players and the strategies are finite.

Theorem 6.2 (Nash (1951)) *Every finite game has a (mixed) Nash equilibrium.*

Proof: Define simplex to be

$$\Delta_k = \{x \in \mathbb{R}^{k+1}_{\geq 0} : \Sigma_{i=1}^{k+1} x_i = 1\}.$$

Clearly, this is a convex and compact set in $\mathbb{R}^{k+1}$. Consider two players (the case with n players is an extension of this idea). Say, player 1 has m strategies labeled $1, \ldots, m$ and player 2 has n strategies labeled

$1, \dots, n$. So, player 1's mixed strategy is a point in Δ_{m-1} and player 2's mixed strategy is a point in Δ_{n-1}. The set of mixed strategy profiles is a point in $\Delta_{m-1} \times \Delta_{n-1}$. Since we are in a two player game, the utilities can be expressed in terms of two matrices A and B, both in $\mathbb{R}^{m \times n}$, denoting the utilities of players 1 and 2 respectively at the pure strategy profiles given by the rows and columns of the matrices. For mixed strategies $p \in \Delta_{m-1}$ and $q \in \Delta_{n-1}$ for players 1 and 2 respectively

$$u_1(p,q) = p^\top Aq, u_2(p,q) = p^\top Bq.$$

Define the following quantities,

$$c_i(p,q) = \max\{A_i q - p^\top Aq, 0\}, \text{ where } A_i \text{ is the } i^{th} \text{ row of } A, \ i \in \{1, \dots, m\}.$$
$$d_j(p,q) = \max\{p^\top B_j - p^\top Bq, 0\}, \text{where } B_j \text{ is the } j^{th} \text{ column of } B, \ j \in \{1, \dots, n\}.$$

Clearly, both quantities are non-negative for all i, j.

Now, we define two functions P and Q as follows

$$P_i(p,q) = \frac{p_i + c_i(p,q)}{1 + \sum_{k=1}^m c_k(p,q)}, \ i \in \{1, \dots, m\}; \qquad Q_j(p,q) = \frac{q_j + d_j(p,q)}{1 + \sum_{k=1}^n d_k(p,q)}, \ j \in \{1, \dots, n\}.$$

Clearly, $P_i(p,q) \geq 0, \forall i$ and $\sum_{i=1}^m P_i(p,q) = 1$. Hence $P(p,q) \in \Delta_{m-1}$ and similarly we see that $Q(p,q) \in \Delta_{n-1}$. Define the transformation function

$$T(p,q) = (P(p,q), Q(p,q)).$$

We see that, $T : \Delta_{m-1} \times \Delta_{n-1} \mapsto \Delta_{m-1} \times \Delta_{n-1}$, and maps a convex and compact set onto itself. From the definitions it is clear that c_i and d_j's are continuous in (p,q), therefore, P_i's and Q_j's are also continuous which implies that T is continuous. Hence, by Brouwer's fixed point theorem,

$$\exists \ (p^*, q^*) \text{ s.t. } T(p^*, q^*) = (p^*, q^*).$$

Claim 6.3

$$\sum_{k=1}^m c_k(p^*, q^*) = 0; \qquad \sum_{k=1}^n d_k(p^*, q^*) = 0.$$

Proof:[of Claim] Suppose the claim is false, i.e., $\sum_{k=1}^m c_k(p^*, q^*) > 0$. Since (p^*, q^*) is a fixed point of T

$$p_i^* = \frac{p_i^* + c_i(p^*, q^*)}{1 + \sum_{k=1}^m c_k(p^*, q^*)} \Rightarrow p_i^* \left(\sum_{k=1}^m c_k(p^*, q^*) \right) = c_i(p^*, q^*). \tag{6.1}$$

Define a subset of indices as $I = \{i : p_i^* > 0\}$. We see that

$$I = \{i : p_i^* > 0\} = \{i : c_i(p^*, q^*) > 0\} = \{i : A_i q^* > p^{*\top} A q^*\}. \tag{6.2}$$

The first equality follows from eq. (6.1) and our assumption that $\sum_{k=1}^m c_k(p,q) > 0$. The second equality come from the definition of c_i. Define $u_i^* := p^{*\top} A q^*$.

Now we see

$$u_1^* = \sum_{i=1}^m p_i^* A_i q^* = \sum_{i \in I} p_i^* (A_i q^*) > \left(\sum_{i \in I} p_i^* \right) u_1^* = u_1^*.$$

The first equality is by definition, the second inequality holds since p_i^* is positive only on I (by definition), the inequality holds from eq. (6.2), and the last equality holds since u_i^* is a scalar and comes out of the summation. The inequality above is a contradiction. Similarly we can prove the claim for $\sum_k d_k$. Hence our claim is proved. ■

By this claim, $\sum_{k=1}^{m} c_k(p^*, q^*) = 0$. Since $c_k(p^*, q^*) \geq 0, \forall k = 1, \ldots, m$, it implies that $c_k(p^*, q^*) = 0 \forall k = 1, \ldots, m$. By definition of c_i's, we then have

$$A_i q* \leq p^{*\top} A q^*$$
$$\Rightarrow \sum_{i=1}^{m} p'_i A_i q^* \leq p^{*\top} A q^*.$$

The implication holds for any arbitrary mixed strategy p' of player 1. Similarly we can show that q^* is a best response for player 2 against the mixed strategy p^* played by player 1. Therefore (p^*, q^*) is a MSNE. ■