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Lecturer: Swaprava Nath *Scribe(s): Gundeep Arora*

5.1 Recap

In the last lecture, we saw an example where an MSNE existed but PSNE did not. We also looked at the characterization theorem of MSNE. In this lecture, we shall prove the theorem and compute MSNE using this theorem.

5.2 Characterization Theorem of MSNE

For completeness, we restate the theorem from the last lecture. This theorem lists down the essential characteristics of a mixed strategy Nash equilibrium.

Theorem 5.1 *A strategy profile $(\sigma_i^*, \sigma_{-i}^*)$ is a MSNE iff $\forall i \in N$*

1. *$u_i(s_i, \sigma_{-i}^*)$ is same $\forall s_i \in \delta(\sigma_i^*)$*
2. *$u_i(s_i, \sigma_{-i}^*) \geq u_i(s_i', \sigma_{-i}^*) \quad \forall s_i \in \delta(\sigma_i^*), \quad s_i' \notin \delta(\sigma_i^*)$*

Before we prove this theorem, we shall state an observation,

Remark 1 *Clearly, while maximizing the expectation of a random variable with finite support w.r.t. the distribution is achieved when the distribution places the whole probability mass at the maximum value (or splits arbitrarily over the maximum values). Hence*

$$\max_{\sigma_i \in \Delta(S_i)} u_i(\sigma_i, \sigma_{-i}^*) = \max_{s_i \in S_i} u_i(s_i, \sigma_{-i}^*).$$

Moreover, if σ_i^* is Nash equilibrium strategy for player i, then the strategy that maximizes the expected utility $u_i(s_i, \sigma_{-i}^*)$ will always be a part of the support of σ_i^*, $\delta(\sigma_i^*)$. Otherwise, we can create another strategy with all the probability mass on this maximum one that falls outside the support and this strategy would have a strictly better utility for player i, contradicting σ^* being MSNE. Therefore we have

$$\max_{\sigma_i \in \Delta(S_i)} u_i(\sigma_i, \sigma_{-i}^*) = \max_{s_i \in S_i} u_i(s_i, \sigma_{-i}^*) = \max_{s_i \in \delta(\sigma_i^*)} u_i(s_i, \sigma_{-i}^*).$$

Proof: ($\Rightarrow$) We first prove the necessity part, i.e., that given a strategy profile $(\sigma_i^*, \sigma_{-i}^*)$ is an MSNE, the two conditions hold true. Given $(\sigma_i^*, \sigma_{-i}^*)$ is a MSNE

$$\begin{aligned} u_i(\sigma_i^*, \sigma_{-i}^*) &= \max_{\sigma_i \in \Delta(S_i)} u_i(\sigma_i, \sigma_{-i}^*) \\ &= \max_{s_i \in S_i} u_i(s_i, \sigma_{-i}^*) \\ &= \max_{s_i \in \delta(\sigma_i^*)} u_i(s_i, \sigma_{-i}^*) \end{aligned} \tag{5.1}$$

Here the first and second equality follows due to remark 1. Also, by definition of expected utility for the given strategy profile we have

$$\begin{aligned} u_i(\sigma_i^*, \sigma_{-i}^*) &= \sum_{s_i \in S_i} \sigma_i^*(s_i) \cdot u_i(s_i, \sigma_{-i}^*) \\ &= \sum_{s_i \in \delta(\sigma_i^*)} \sigma_i^*(s_i) \cdot u_i(s_i, \sigma_{-i}^*) \end{aligned} \tag{5.2}$$

Equating the 5.1 and 5.2, we see that the expectation and the maximum value of a set are equal. This can happen only when either the set is singleton or all the elements take the same value. This proves the first condition mentioned.

We prove the second condition using the idea of contradiction. Suppose the condition does not hold, i.e.

$$\exists s_i \in \delta(\sigma_i^*), s_i' \notin \delta(\sigma_i^*) \text{ s.t. } u_i(s_i, \sigma_{-1}^*) < u_i(s_i', \sigma_{-i}^*)$$

According to our previous argument, $u_i(s_i, \sigma_{-1}^*)$ is same for all $s_i \in \delta(\sigma_i^*)$. Hence the LHS of the above inequality is equal to $u_i(\sigma_i^*, \sigma_{-i}^*) =: u_i^*$. Choose a strategy σ_i' for player i, such that

$$\begin{aligned} &\sigma_i'(s_i') = 1 \\ &\sigma_i'(s_i) = 0, \forall s_i \in S_i \setminus \{s_i'\} \end{aligned} \tag{5.3}$$

Using this mixed strategy, we compute the expected utility for player i at the strategy profile $(\sigma_i', \sigma_{-i}^*)$ and find

$$u_i(\sigma_i^*, \sigma_{-i}^*) < u_i(\sigma_i', \sigma_{-i}^*).$$

The above inequality contradicts the fact that $(\sigma_i^*, \sigma_{-i}^*)$ is an MSNE. This proves our second condition as well.

($\Leftarrow$) To prove the sufficiency, we assume that given the two conditions of the characterization theorem hold. We define $u_i(s_i, \sigma_{-i}^*) =: m_i(\sigma_{-i}^*)$, for all $s_i \in \delta(\sigma_i^*)$. This is possible to define due to condition (1). Similarly, using (2), we conclude $m_i(\sigma_{-i}^*) = \max_{s_i \in S_i} u_i(s_i, \sigma_i^*)$.

$$\begin{aligned} u_i(\sigma_i^*, \sigma_{-i}^*) &= \sum_{s_i \in \delta(\sigma_i^*)} \sigma_i^*(s_i) \cdot u_i(s_i, \sigma_{-i}^*) \\ &= m_i(\sigma_{-i}^*) \\ &= \max_{s_i \in S_i} u_i(s_i, \sigma_i^*) \\ &= \max_{\sigma_i \in \Delta(S_i)} u_i(\sigma_i, \sigma_{-1}^*) \\ &\geq u_i(\sigma_i, \sigma_{-1}^*) \quad \forall \sigma_i \in \Delta(S_i) \end{aligned} \tag{5.4}$$

The first equality holds by definition of $\delta(\sigma_i^*)$. The next two equalities hold due to conditions (1) and (2) as explained above. The last equality is by remark 1 Hence the strategy profile $(\sigma_i^*, \sigma_{-i}^*)$ is a MSNE. ■

5.3 Some Examples for MSNE

With the proof of the characterization theorem done, we shall now look at some examples with the lens of that theorem and argue whether possible strategy profiles are MSNE or not based on their satisfiability of the two conditions of the theorem.

5.3.1 The Matching Coins Game / Penalty Shoot-out Game

In the penalty shoot-out game, that we looked at last time, we had concluded that no pure strategy Nash equilibrium existed for that game. The game poses the utility functions of either player hitting or missing the target.

1 \ 2	H	T
H	(1,-1)	(-1,1)
T	(-1,1)	(1,-1)

Let the probability of player 1 choosing H be p. Similarly, for player 2, let that probability be q. We shall now look at some of the strategy profiles and argue if it can be an MSNE or not.

- k =**({H},{H})** : In this case, $u_1(k) = 1$ and $u_2(k) = -1$. The for the given choice of strategy for player 1, the player 2, can choose {H} and have a utility $u_2\{\text{H}\}, \{\text{H}\}) = 1$, which is more than $u_2(k)$. We can establish the same result for each of the other pure strategy profiles will not be a MSNE. This is just what we had seen in the previous lecture, where we had concluded that no pure strategy Nash equilibrium existed for this game. These pure strategy profiles also do not satisfy the second condition of the characterization theorem.
- k =**({H},{H,T})** : In this case, $u_1(k) = 1$ and

$$\begin{aligned} u_2(\{\text{H}\}, \{\text{H}\}) &= -1 = 0 \\ u_2(\{\text{H}\}, \{\text{T}\}) &= 1 \\ &= 2p - 1 \end{aligned} \tag{5.5}$$

  For this to be a MSNE, by the first condition of the characterization equation, we have

$$\begin{gathered} u_2(\{\text{H}\}, \{\text{H}\}) = u_2(\{\text{H}\}, \{\text{T}\}) \\ -1 \neq 1 \end{gathered} \tag{5.6}$$

  This hence cannot be a MSNE. By using the argument of symmetry for this game, we can argue that any strategy profile that has a pure strategy for any user will not be a Nash equilibrium.
- k =**({H,T},{H,T})** : In this case,

$$\begin{aligned} u_2(\{\text{H,T}\}, \{\text{H}\}) &= (-1) \cdot p + 1 \cdot (1 - p) \\ &= 1 - 2p \\ u_2(\{\text{H,T}\}, \{\text{T}\}) &= 1 \cdot p + (-1) \cdot (1 - p) \\ &= 2p - 1 \\ u_1(\{\text{H}\}, \{\text{H,T}\}) &= 1 \cdot q + (-1) \cdot (1 - q) \\ &= 2q - 1 \\ u_1(\{\text{T}\}, \{\text{H,T}\}) &= (-1) \cdot q + 1 \cdot (1 - q) \\ &= 1 - 2q \end{aligned} \tag{5.7}$$

For this strategy for this to be a valid MSNE, from the characterization theorem that we just proved, we have the following,

$$\begin{aligned} u_2(\{\text{H,T}\},\{\text{H}\}) &= u_2(\{\text{H,T}\},\{\text{T}\}) \\ 1-2q &= 2q-1 \\ q &= \frac{1}{2} \\ u_1(\{\text{H}\},\{\text{H,T}\}) &= u_1(\{\text{T}\},\{\text{H,T}\}) \\ 2p-1 &= 1-2p \\ p &= \frac{1}{2} \end{aligned} \tag{5.8}$$

Since, the system of equations have a solution, the the strategy profile $(\{\text{H}:\frac{1}{2},\text{T}:\frac{1}{2}\},\{\text{H}:\frac{1}{2},\text{T}:\frac{1}{2}\})$ is a MSNE. The first condition is vacuously satisfied.

5.3.2 Game-Selection Problem

We also looked at the problem, where the two friends who had different preferences towards going for a game while still going together to have a good time. More formally, the game with the utility function is represented as :

1 \ 2	F	C
F	(2,1)	(0,0)
C	(0,0)	(1,2)

Like the previous example, let the probability of player 1 choosing H be p. Similarly, for player 2, let that probability be q. We shall now look at some of the strategy profiles and argue if it can be an MSNE or not.

- **Pure strategy** : Consider a pure strategy, k =({F},{F}). For this strategy profile, we have $u_1(k)=2$ and $u_2(k)=1$. This is a PSNE, hence definitely a MSNE. As we had seen in the previous lecture, the profile k =({C},{C}) is also a MSNE. Both these profiles trivially satisfy the two conditions of the characterization theorem.

- **({F},{F,C})** : For this strategy profile,

$$\begin{aligned} u_2(\{\text{F}\},\{\text{F}\}) &= 1 \\ u_2(\{\text{F}\},\{\text{C}\}) &= 0 \end{aligned} \tag{5.9}$$

  Clearly the above equations state that the first condition to be is violated and hence this profile is not a MSNE.

- **({C},{F,C})** : For this strategy profile,

$$\begin{aligned} u_2(\{\text{C}\},\{\text{F}\}) &= 0 \\ u_2(\{\text{C}\},\{\text{C}\}) &= 2 \end{aligned} \tag{5.10}$$

  Clearly the above equations state that the first condition to be is violated and hence this profile is not a MSNE.

- **({F,C},{F})** : For this strategy profile,

$$\begin{aligned} u_1(\{\text{F}\},\{\text{F}\}) &= 2 \\ u_1(\{\text{C}\},\{\text{F}\}) &= 0 \end{aligned} \tag{5.11}$$

  Clearly the above equations state that the first condition to be is violated and hence this profile is not a MSNE.

- **({F,C},{C})** : For this strategy profile,

$$\begin{aligned} u_1(\{\text{C}\},\{\text{C}\}) &= 1 \\ u_1(\{\text{F}\},\{\text{C}\}) &= 0 \end{aligned} \tag{5.12}$$

  Clearly the above equations state that the first condition to be is violated and hence this profile is not a MSNE.

- **({F,C},{F,C})** : In this case,

$$\begin{aligned} u_2(\{\text{F,C}\},\{\text{F}\}) &= 1 \cdot p + 0 \cdot (1-p) \\ &= p \\ u_2(\{\text{F,C}\},\{\text{C}\}) &= 0 \cdot p + 2 \cdot (1-p) \\ &= 2 - 2p \\ u_1(\{\text{F}\},\{\text{F,C}\}) &= 2 \cdot q + 0 \cdot (1-p) \\ &= 2q \\ u_1(\{\text{C}\},\{\text{F,C}\}) &= 0 \cdot q + 1 \cdot (1-q) \\ &= 1 - q \end{aligned} \tag{5.13}$$

  For this strategy for this to be a valid MSNE, from the characterization theorem that we just proved, we have the following,

$$\begin{aligned} u_2(\{\text{F,C}\},\{\text{F}\}) &= u_2(\{\text{F,C}\},\{\text{C}\}) \\ p &= 2 - 2p \\ p &= \frac{1}{3} \\ u_1(\{\text{F}\},\{\text{F,C}\}) &= u_1(\{\text{C}\},\{\text{F,C}\}) \\ 2q &= 1 - q \\ p &= \frac{1}{3} \end{aligned} \tag{5.14}$$

  Since, the system of equations have a solution, the the strategy profile $(\{\text{F} : \frac{1}{3}, \text{T} : \frac{2}{3}\}, \{\text{F} : \frac{2}{3}, \text{C} : \frac{1}{3}\})$ is a MSNE. The first condition is vacuously satisfied.

This problem can be extended to have the following form,

1 \ 2	F	C	D
F	(2,1)	(0,0)	(1,1)
C	(0,0)	(1,2)	(2,0)

For such a case, a mixed strategy for player two, will generally look like, $\{\mathrm{F} : \mathrm{q}_1; \mathrm{C} : \mathrm{q}_2; \mathrm{D} : 1 - \mathrm{q}_1 - \mathrm{q}_2\}$. Proceeding similar to the original example, we have,

$$\begin{aligned} u_1(\{\text{F}\}, \{\text{F,C,D}\}) &= u_1(\{\text{C}\}, \{\text{F,C,D}\}) \\ 2 \cdot q_1 + 0 \cdot q_2 + 1 \cdot (1 - q_1 - q_2) &= 0 \cdot q_1 + 1 \cdot q_2 + 2 \cdot (1 - q_1 - q_2) \\ 1 + q_1 - q_2 &= 2 - 2q_1 - q_2 \\ q_1 &= \frac{1}{3} \end{aligned} \tag{5.15}$$

However, we do not have any specific value for q_2, so any value of q_2 between $[0, \frac{2}{3}]$ will yield an MSNE. The solution for the player 1 remains the same.

5.4 General Principle for finding MSNE

As we saw in the two examples above, in order to evaluate the Nash equilibrium. we enumerated all the possible supports of the Cartesian product, $S_1 \times S_2 \times \cdots \times S_n$, where S_i is the set of all possible options for player i and then used the two conditions to check if it was a MSNE or not. This means that the number of supports, K, that need to be enumerated is,

$$K = (2^{|S_1|} - 1) \times (2^{|S_2|} - 1) \times \cdots \times (2^{|S_n|} - 1)$$

More formally put, the problem that we are trying to solve here is, given a support $X_i \subseteq S_i$ and a support profile $X_1 \times X_2 \times \cdots X_n$, we have from the first condition of the characterization theorem, i.e., $u_i(s_i, \sigma_{-i})$ should be same for all $s_i \in \delta(\sigma_i)$

$$w_i = \sum_{s_{-i} \in S_{-i}} \left(\prod_{j \neq i} \sigma_j(s_j) \right) u_i(s_i, s_{-i}), \forall s_i \in X_i, \forall i \in N.$$

From the first and second conditions together, we get

$$w_i \geq \sum_{s_{-i} \in S_{-i}} \left(\prod_{j \neq i} \sigma_j(s_j) \right) u_i(s_i, s_{-i}), \forall s_i \in S_i \setminus X_i, \forall i \in N.$$

These equalities and inequalities constitute a feasibility program with variables $w_i, i \in N$, $\sigma_j(s_j), s_j \in S_j, j \in N$. The $\sigma_j(s_j)$'s must satisfy $\sigma_j(s_j) \geq 0, s_j \in S_j, j \in N$ and $\sum_{s_j \in S_j} \sigma_j(s_j) = 1, \forall j \in N$. This is a linear programming problem iff $n = 2$. For $n > 2$, the first set of equality and inequalities are non-linear. To find an MSNE, a brute-force algorithm is to list these set of equalities and inequalities for every support profile in K and solve them. Note that K is exponential in the size of the strategy spaces. Unfortunately, the problem of finding Nash equilibrium is PPAD complete [DGP08], which implies that it is unlikely to have a better algorithm to find MSNE in general games.

Finding Nash equilibrium and its complexity is an active area of research. For two player zero-sum game, we can have a linear program, the equalities to be solved turn out to be simple and hence we have efficient algorithms for it. There are algorithms like Lemke-Howson Algorithms that are proven to be optimal for two player non-zero sum games. We shall look at the existence of the Nash equilibrium in the next lecture.

References

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[DGP08] Daskalakis, Constantinos, Paul W. Goldberg, and Christos H. Papadimitriou. "The complexity of computing a Nash equilibrium." SIAM Journal on Computing 39.1 (2009): 195-259.