CS-698W: Game Theory and Collective Choice	Jul-Nov 2017

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Disclaimer: *These notes aggregate content from several texts and have not been subjected to the usual scrutiny deserved by formal publications. If you find errors, please bring to the notice of the Instructor at swaprava@cse.iitk.ac.in.*

Definition 4.1 (Best response set) *A best response of agent i against the strategy profile s_{-i} of the other players is a strategy that gives the maximum utility against the s_{-i} chosen by other players, i.e.,*

$$B_i(s_{-i}) \;=\; \{s_i \in S_i : u_i(s_i, s_{-i}) \geq u_i(s_i', s_{-i}),\ \forall s_i' \in S_i\}.$$

Observe: If (s_i^*, s_{-i}^*) is a Pure Strategy Nash Equilibrium, then $s_i^* \in B_i(s_{-i}^*)\ \forall i \in N$.

We know that an SDSE is a WDSE. To observe the relation between WDSE and PSNE, we recap the definition of WDSE. To define a WDSE, we need the definition of *Weakly Dominant Strategy* (WDS).

Definition 4.2 (Weakly Dominant Strategy) s_i^* *is WDS if*

1. $u_i(s_i^*, s_{-i}) \geq u_i(s_i', s_{-i}),\ \forall s_i' \in S_i,\ \forall s_{-i} \in S_{-i}$
2. $u_i(s_i^*, \overline{s_{-i}}) > u_i(s_i', \overline{s_{-i}}),\ \forall s_i' \in S_i,$ *for some* $\overline{s_{-i}} \in S_{-i}$

It is important to note that when s_i dominates all s_i' in the definition above, the profile of other players on which the strict inequality holds, can be different for different s_i'. Here is an example illustrating this fact. Note here $u_1(D, D) > u_1(A, D)$ and $u_1(D, A) > u_1(S, A)$. D is WDS for P_1.

P_1 \ P_2	A	D
A	5, 5	0, 5
D	5, 0	1, 1
S	4, 0	1, 1

Table 4.1: Example game to illustrate WDS.

A strategy profile (s_i^*, s_{-i}^*) is a WDSE if s_i^* is a WDS for every $i \in N$. Clearly, a WDSE is a PSNE.

However, in a finite game, even PSNE is not guaranteed to exist. Table 4.2 gives an example. Hence, we arrive at a further weak equilibrium concept named *Mixed Strategy Nash Equilibrium* (MSNE).

P_1 \ P_2	H	T
H	+1, −1	−1, +1
T	−1, +1	+1, −1

Table 4.2: Matching Coins Game

4.1 Mixed Strategy Nash Equilibrium

For a finite set A, $\Delta(A)$ is defined as the set of all probability distributions over A, $\Delta(A) = \{p \in [0,1]^{|A|} : \sum_{a \in A} p(a) = 1\}$. Then $\sigma_i \in \Delta(S_i)$ is a **mixed strategy** of player i, where S_i is their finite strategy set. Mixed strategy is a distribution σ_i over the strategies in S_i , i.e., $\sigma_i : S_i \mapsto [0,1]$ s.t. $\sum_{s_i \in S_i} \sigma_i(s_i) = 1$.

Utility of player i at a mixed strategy profile (σ_i, σ_{-i}) is

$$u_i(\sigma_i, \sigma_{-i}) = \sum_{s \in S} \left(\prod_{i \in N} \sigma_i(s_i) \right) u_i(s_i, s_{-i}),$$

where $s = (s_1, \ldots, s_n)$ and $S = S_1 \times \cdots \times S_n$.

Consider the game as given in Table 4.2. Now suppose Player 1 plays the mixed strategy H with probability p and Player 2 plays H with probability q.

Then the utility u_1 of the player 1 is $u_1((p, 1-p), (q, 1-q))$
$= pq\ u_1(H,H) + p(1-q)\ u_1(H,T) + (1-p)q\ u_1(T,H) + (1-p)(1-q)\ u_1(T,T)$

For a mixed strategy profile $\sigma' = ((1,0), (\frac{1}{2}, \frac{1}{2}))$, $u_1(\sigma') = 1.\frac{1}{2}(+1) + 1.\frac{1}{2}(-1) = 0$

When player i plays pure strategy while all others play mixed strategy, we denote the utility of the player by

$$u_i(s_i, \sigma_{-i}) := \sum_{s_{-i} \in S_{-i}} \left(\prod_{j \neq i} \sigma_j(s_j) \right) u_i(s_i, s_{-i}).$$

Definition 4.3 (Mixed Strategy Nash Equilibrium) *MSNE is a mixed strategy profile* $(\sigma_i^*, \sigma_{-i}^*)$ *s.t.*

$$u_i(\sigma_i^*, \sigma_{-i}^*) \geq u_i(\sigma_i', \sigma_{-i}^*)\ \forall \sigma_i' \in \Delta(S_i),\ \ \forall i \in N.$$

One can define a best response set in terms of mixed strategies in a similar spirit and observe that

$$B_i(\sigma_{-i}) = \{\sigma_i \in \Delta(S_i) : u_i(\sigma_i, \sigma_{-i}) \geq u_i(\sigma_i', \sigma_{-i}),\ \forall \sigma_i' \in \Delta(S_i)\},$$

and if $(\sigma_i^*, \sigma_{-i}^*)$ is a MSNE, then $\sigma_i^* \in B_i(\sigma_{-i}^*), \forall i \in N$.

Now since we have seen all the important equilibrium concepts, fig. 4.1 shows how one equilibrium implies another and thereby the Venn-diagram of the different equilibria. Each of the subset implication in this figure is strict. It is easy to construct examples to show the strictness.

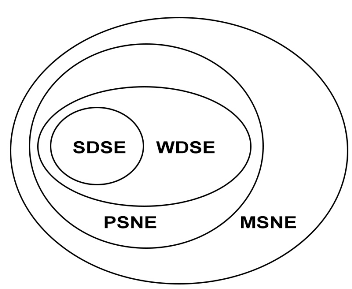

Figure 4.1: Types of Equilibrium

4.2 Computation of MSNE

To compute an MSNE, we first state a result that helps in formulating the problem of finding the equilibrium. To do this, we define the support of a mixed strategy as follows.

Definition 4.4 (Support of a Mixed Strategy) *The support of a mixed strategy σ_i is the subset of the strategy space of i on which the mixed strategy σ_i has positive mass, and is denoted by*

$$\delta(\sigma_i) = \{s_i \in S_i : \sigma_i(s_i) > 0\}.$$

Theorem 4.5 (Characterization of a MSNE) *A mixed strategy profile $(\sigma_i^*, \sigma_{-i}^*)$ is a MSNE iff $\forall i \in N$*

1. *$u_i(s_i, \sigma_{-i}^*)$ is the same for all $s_i \in \delta(\sigma_i^*)$, and*
2. *$u_i(s_i, \sigma_{-i}^*) \geq u_i(s_i', \sigma_{-i}^*), \ \forall \ s_i \in \delta(\sigma_i^*), s_i' \notin \delta(\sigma_i^*)$.*