CS698W: Game Theory and Collective Choice **Jul-Nov 2017**

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Disclaimer: *These notes aggregate content from several texts and have not been subjected to the usual scrutiny deserved by formal publications. If you find errors, please bring to the notice of the Instructor at swaprava@cse.iitk.ac.in.*

3.1 Normal Form Representation

We revisit the game of neighbouring kingdoms' dilemma. This is a one-shot non-cooperative game. The outcomes (o_i) in this game are as follows: (A,A)= o_1, (A,D)=o_2, (D,A)=o_3, (D,D)=o_4.

1\2	A	D
A	5,5	0,6
D	6,0	1,1

A *normal form* or *strategic form* representation of a game is given by the tuple $\langle N, (S_i)_{i\in N}, (u_i)_{i\in N}\rangle$, where $N = \{1, 2, \dots, n\}$ is the set of players, S_i is the set of strategies of player i, and $u_i : S_1 \times \cdots \times S_n \mapsto \mathbb{R}$ is the vNM utility function of agent i. We will denote a specific strategy of player i by $s_i \in S_i$. The strategy profile of all agents except player i is denoted by s_{-i}. A strategy profile is the tuple $s := \{s_1, s_2, \dots, s_n\}$ which is also represented by (s_i, s_{-i}). Note that $s \in S_1 \times \cdots \times S_n = S$. Hence, utility of player i is $u_i(s_1, s_2, \dots, s_n)$, where s_j represents strategy picked by player j, $\forall j \in N$.

Difference between Strategy and Action We distinguish action from strategy as follows. An *action* is the smallest available choice a player has in a game. For instance, it is either to pick action Agri or Def in the game above. But a *strategy* is a more complicated object than an action, since it may involve a combination of actions, or a mixture of actions. For instance, in a multi-round game, a strategy will be a *complete contingent plan* of which action to play at which stage and state of the game. However, in the one-shot non-cooperative game example above, the strategy is to pick an action, and therefore the set of strategies is same as the set of actions. But this distinction is worth remembering.

3.2 Behaviour of players

To predict an outcome of a game, we need some behavioral assumptions on the players. We assume following behaviours of the players :

- **Rationality:** Every player picks strategy to maximize her utility.
- **Intelligence:** Every player possesses enough information about the game and is able to find the best strategy for her.

Assumption 3.1 *Every player is rational and intelligent.*

Common Knowledge: A fact is known as a common knowledge if

1. All players know the fact, and
2. All players know that all other players know the fact, and
3. All players know that all other players know that all other players know the fact, and ... ad infinitum.

Example of common knowledge: Consider an isolated island where lives three blue-eyed individuals (eyes can be either blue or black). The individuals do not talk to each other but can listen and the island has no reflecting media, e.g., mirrors etc., where they can see their own eye color. One day a sage comes to the island and says "Blue-eyed people are bad for the island and must leave. There is at least one blue-eyed person in this island". Assume that the sage's statements cannot be disputed. Also, if a person realizes that his eye color is blue, he leaves at the end of the day. Let us see the implication of this statement and how common knowledge percolates to the outcome.

If there were only one blue-eyed person, he would have seen that the other two had black eyes, realized that his eye color is blue (since sage is always correct), leaves at the end of day one. Every other player understands this.

If there were two blue-eyed persons, then both of them will see one blue and one black eyed person, hope that he is not the blue eyed one and wait until the second day if the other blue-eyed person leaves on day one. When it does not happen, he realizes that both of them had blue eyes, so they both leave at the end of day two. Every player understands this.

Since there are three blue-eyed persons, then extending the same argument, we see that every player will wait till day three if anyone leaves. When nobody left on day two, it becomes clear that all of them had blue eyes, and they all leave at the end of day three.

Assumption 3.2 *The fact that all players are rational and intelligent is a Common Knowledge.*

3.3 Some important definitions

- **Strictly dominated strategy:** A strategy s_i' is strictly dominated by s_i, if $\forall s_{-i} \in S_{-i}$
$$u_i(s_i, s_{-i}) > u_i(s_i', s_{-i}).$$
- **Weakly dominated strategy:** A strategy s_i' is weakly dominated by s_i, if $\forall s_{-i} \in S_{-i}$
$$u_i(s_i, s_{-i}) \geq u_i(s_i', s_{-i}),$$
and $\exists\ s_{-i} \in S_{-i}$ such that
$$u_i(s_i, s_{-i}) > u_i(s_i', s_{-i}).$$
- **Strictly/Weakly dominant strategy:** A strategy s_i is strictly/weakly dominant strategy of player i if s_i strictly/weakly dominates all other $s_i' \in S_i \setminus \{s_i\}$
- **Strictly/Weakly dominant strategy equilibrium:** A strategy profile (s_i^*, s_{-i}^*) is an SDSE/WDSE if s_i^* is a SDS/WDS for every $i, i \in N$.

Does every Game have a SDSE/WDSE? Can there be more than one Nash equilibrium? We will answer these questions through the following example.

1\2	C	F
C	(2,1)	(0,0)
F	(0,0)	(1,2)

This game does not have any SDSE or WDSE. Hence, it is clear that every game is not guaranteed to have a SDSE/WDSE. Hence we come to a weaker notion of equilibrium which is called the Nash equilibrium.

Definition 3.3 (Pure strategy Nash equilibrium) *A strategy profile* (s_i^*, s_{-i}^*) *is a* pure strategy Nash equilibrium *if* $\forall i \in N$ *and* $\forall s_i \in S_i$

$$u_i(s_i^*, s_{-i}^*) \geq u_i(s_i, s_{-i}^*).$$

We see that there exists two pure strategy Nash equilibria in the game above: (C,C) and (F,F). However when there exists an SDSE, there is exactly one equilibrium.