Lecture 2: August 2, 2017

Lecturer: Swaprava Nath *Scribe(s): Dhawal Upadhyay*

2.1 Introduction

Game theory is the formal study of strategic interaction among multiple agents/players, that have individual utilities/payoffs to maximize. In analyzing a game, we may sometimes encounter counter-intuitive results. First, the stage is set with the discussion of one such problem – the Neighboring Kingdoms' dilemma. Then we discuss discuss preference relations, and the requirements for the preference relation to have a utility representation.

2.2 The Neighboring Kingdoms' Dilemma

Suppose there are two kingdoms, each having limited resources. The kingdoms have been hostile for some time, and thus need to boost their defense. Meanwhile, they are also starving, and need to improve their agriculture. Due to limited resources, they can only do one of the two things – boost defense or agriculture.

Let's call the kingdoms **A** and **B**. Kings of A and B need to make a call between the two options. They don't know what the other king will choose, and they cannot coordinate. We will call each kingdom *players* in this game. Consider the following payoff table. The rows represent the actions of player A and columns represent the actions of player B. The entries in the matrix are tuples that represent the utilities of the two players.

A\B	Agri	Def
Agri	5,5	0,6
Def	6,0	1,1

Let us say the numbers represent the happiness quotient of the two kingdoms. If both A and B choose agriculture, they both are happy. Thus they get high utilities! If A chooses agriculture and B defense, then B will attack A and loot their resources (A cannot attack because of B's defense). So B gets 6, and A gets 0. Same happens the other way around. When both choose defense, then although they may not have much to eat, but are still content that nobody can loot them! Thus the utility profile is $(1, 1)$. What should the players choose?

Consider this problem from A's perspective. If B chooses Agri, then A is better of choosing Def, since the quotient he gets is 6 (compared to 5 if he chooses Agri).
If B chooses Def, then A is better off choosing Def, since the quotient in this case is 1 (compared to 0 if he chooses Agri).
So no matter whether B chooses Agri or Def, A is better of choosing Def. Same happens other way around, since the problem is symmetric from B's perspective.

Thus both end up choosing Def, and get a quotient of 1 each. If they were allowed to cooperate, they could have chosen Agri each, leading to a better quotient of 5 each.

A similar version of this problem is the Prisoner's Dilemma

2.3 Notation

A set of players $N = \{1, \ldots, n\}$.
The set of actions of player i is denoted by A_i, a specific action is denoted by a_i, $a_i \in A_i$.
Utility/payoff of agent i: $u_i = A_1 \times A_2 \times \cdots \times A_n \mapsto \mathbb{R}$.
The cartesian product of the action sets will also be represented by $A := \times_{i \in N} A_i$. An element of A is called an *action profile* and is represented by the notations $a \in A$ or $(a_i, a_{-i}) \in A$. The notation a_{-i} represents the action profile of all the players except player i.

When all players choose their respective actions, $a_1, \ldots, a_n$, we say that an *outcome* has realized. Hence, an action profile can be considered as an outcome, say o_ℓ. In the example above, there are four outcomes of the game: (Agri, Agri), (Agri, Def), (Def, Agri), (Def, Def). A *lottery* over the outcomes is a probability distribution over these outcomes, and will be denoted by $[p_1 : o_1, p_2 : o_2, \ldots, p_k : o_k]$, where p_i is the probability of occurance of outcome o_i. Denote the set of outcomes by O.

A relation that signifies the preferences of a player/agent over different outcomes is called a *preference relation*. A preference relation $\succeq$ is a subset of $O \times O$. We will denote o_1 to be at least as preferred as o_2 by $o_1 \succeq o_2$ – which also includes the case that the agent is indifferent between o_1 and o_2. To make the distinction between strict preference and indifference, we will use the notation $o_1 \succ o_2$ which is same as "$o_1 \succeq o_2$ but *not* $o_2 \succeq o_1$" and $o_1 \sim o_2$ which is same as "$o_1 \succeq o_2$ *and* $o_2 \succeq o_1$. In the example above, we have encoded the preference using a utility function, for example, $u_1(o_1) = u_1(\text{Agri, Agri}) = 5$. But it is important to remember that not every preference can be encoded into an utility representation. Consider the following example.

2.3.1 Preference relation without utility representation

Consider a family of three people – child, father, and mother. Three food items a, b, c.
Preference ordering:
Child : $a \succ b \succ c$
Father : $b \succ c \succ a$
Mother : $c \succ a \succ b$
Let's we consider the majority preference order, i.e., the order of items by the majority of the family. If we compare a and b, child and father prefer a, while only mother prefers b. So a is preferred over b by the majority. Similarly, if we do this for pairs (b, c), we will see b is preferred over c by majority. But while comparing (c, a), we will arrive at c being more preferred to a. Hence, we get a cycle ($a \succ_{\text{maj}} b \succ_{\text{maj}} c \succ_{\text{maj}} a$). Clearly, this preference relation does not have a utility representation, as one cannot assign real values that satisfy these inequalities.

Hence, we discuss the conditions that are sufficient for a preference relation to have a utility representation.

2.4 von-Neumann-Morgenstern utility theorem

Axioms for utility representation:

1. **Completeness**: $\forall o_1, o_2 \in O$, exactly one of the following holds: $o_1 \succ o_2$, or $o_2 \succ o_1$ or $o_1 \sim o_2$.

2. **Transitivity**: If $o_1 \succ o_2$ and $o_2 \succ o_3$ then $o_1 \succ o_3$.

3. **Substitutability**: If $o_1 \sim o_2$, then for every sequence $o_3, \ldots, o_k$, and probability masses $p, p_3, \ldots, p_k \in [0, 1]$, s.t. $p + \sum_{i=2}^{n} p_i = 1$, the following holds: $[p : o_1, p_3 : o_3, \ldots, p_k : o_k] \sim [p : o_2, p_3 : o_3, \ldots, p_k : o_k]$.

4. **Decomposability**: For lotteries l_1, l_2, if $P_{l_1}(o_i) = P_{l_2}(o_i)$, $\forall o_i \in O$, then $l_1 \sim l_2$.
   Example:
   $l_1 = [0.5 : [0.4 : o_1, 0.6 : o_2], 0.5 : o_3]$
   $l_2 = [0.2 : o_1, 0.3 : o_2, 0.5 : o_3]$
   Here $l_1 \sim l_2$.

5. **Monotonicity**: If $o_1 \succ o_2$, and $1 > p > q > 0$, then
   $[p : o_1, 1 - p : o_2] \succ [q : o_1, 1 - q : o_2]$.

6. **Continuity**: If $o_1 \succ o_2$ and $o_2 \succ o_3$, then $\exists p \in [0, 1]$ s.t. $o_2 \sim l_2 = [p : o_1, 1 - p : o_3]$.

Theorem 2.1 (von-Neumann, Morgenstern (1944)) *If a preference relation $\succeq$ satisfies Axioms 1 to 6, then $\exists u : O \mapsto [0, 1]$ s.t.*

1. $u(o_1) \geqslant u(o_2) \iff o_1 \succeq o_2$,

2. $u([p_1 : o_1, p_2 : o_2, \ldots, p_k : o_k]) = \sum_{i=1}^{k} (p_i u(o_i))$.

Quite naturally, these utilities are called von-Neumann-Morgenstern (vNM) utilities.

2.5 Summary

We studied six axioms that ensure the existence of a utility representation of a preference relation. These axioms are quite intuitive, and the proof can be found in any standard text (e.g., see [SLB08]). In many real life situations, these axioms are always satisfied, and henceforth, we shall assume the existence of a utility function for any preference that we discuss.

References

[SLB08] Shoham, Yoav, and Kevin Leyton-Brown. "Multiagent systems: Algorithmic, game-theoretic, and logical foundations". Cambridge University Press, 2008.