CS698W: Topics in Game Theory and Collective Choice

Problem Set 2
Course Homepage: `https://swaprava.wordpress.com/game-theory-collective-choice/`

1. Consider a two agent model with three alternatives $\{a, b, c\}$. Table 1 shows two preference profiles of preferences. Suppose $f(P_1, P_2) = a$. Show that if f is strategyproof then $f(P'_1, P'_2) = b$. You are allowed to use the result that for any preference profile $(\bar{P}_1, \bar{P}_2)$, $f(\bar{P}_1, \bar{P}_2) \in \{\bar{P}_1(1), \bar{P}_2(1)\}$ (but do not use any other result from the lectures).

P_1	P_2	P'_1	P'_2
a	c	b	c
b	b	a	a
c	a	c	b

Table 1: Two Preference Profiles

2. Consider the unanimous SCF f defined as follows. If $P_1(1) = \ldots = P_n(1) = a$, then $f(P_1, \ldots, P_n) = a$. Else, $f(P_1, \ldots, P_n) = b$ for some alternative $b \in A$. In other words, f satisfies unanimity wherever possible and picks a "status-quo" alternative b otherwise. Is f strategyproof? If yes, prove it, if not, provide a counterexample.

3. Let A be a finite set of alternatives and $f : \mathcal{P}^n \to A$ be a social choice function that is unanimous and strategyproof. Suppose $|A| \geqslant 3$.

   Now, consider another social choice function $g : \mathcal{P}^2 \to A$ defined as follows. The SCF g only considers profiles of two agents, denote these two agents as 1 and 2. For any $(P_1, P_2) \in \mathcal{P}^2$, let
$$g(P_1, P_2) = f(P_1, P_2, P_1, P_1, \ldots, P_1),$$
   i.e., the outcome of g at (P_1, P_2) coincides with the outcome of f at the profile where agents 1 and 2 have types P_1 and P_2 respectively, and all other agents have type P_1.

   Show that g is a dictatorial SCF. [Hint: you may use the Gibbard-Satterthwaite theorem.]

4. Let the number of alternatives be m. Find the number of single-peaked strict preference orderings with respect to $<$ (an exogenously given ordering of alternatives). [Hint: for a fixed peak, how many strict orderings are possible in the single-peaked domain?]

5. Consider the single-peaked domain model. A social choice function f is manipulable by a group of agents $K \subseteq N$ if for some preference profile (P_K, P_{-K}) there exists some preference profile P'_K of agents in K such that $f(P'_K, P_{-K}) P_i f(P_K, P_{-K})$ for all $i \in K$. A social choice function f is **group strategyproof** if cannot be manipulated by any group of agents. Is the median voter SCF group strategyproof? Explain your answer.

6. Let $A = [0, 1]$ and assume that agents have single peaked preferences over $A = [0, 1]$. Consider the following social choice function.

Definition 1 *A social choice function f is a* **generalized median voter** *social choice function if there exists weights y_S for every $S \subseteq N$ satisfying*

(a) $y_\emptyset = 0$, $y_N = 1$ *and*

(b) $y_S \leq y_T$ *for all* $S \subseteq T$

such that for all preference profile P, $f(P) = \max_{S \subseteq N} z(S)$, where $z(S) = \min\{y_S, P_i(1) : i \in S\}$.

Show that a generalized median voter SCF is strategyproof in the single-peaked domain.

7. In the private divisible good allocation model, discuss a social choice function that is strategy-proof and Pareto efficient but not anonymous.

8. A seller is selling an object to an agent whose value (type) for the object lies in the interval $I \equiv [0, 1]$. The seller uses an allocation rule $f : I \mapsto [0, 1]$ and a payment rule $p : I \mapsto \mathbb{R}$. Denote the mechanism (f, p) as M.

   Fix an $\epsilon \in (0, 1]$. The mechanism M satisfies only **local** incentive compatibility constraints: for every $t \in I$ and for every $s \in I$ such that $|s - t| \leqslant \epsilon$

   $$tf(t) - p(t) \geqslant tf(s) - p(s).$$

   Show that M is (global) dominant strategy incentive compatible. [**Note:** this requires that both the allocation and the payment rule should satisfy the DSIC condition for any arbitrary t and t' in that interval.]

9. Consider a simplified setting of sponsored search auction with the click-through rate depending only on the position – hence the value of an advertiser i placed in position j is

   $$v_{ij} = pos_j \cdot v_i.$$

   Conventionally, we assume $pos_j > pos_{j+1}$.

   Definition 2 *An allocation x is* **locally envy-free** *if there exists prices $\{p_j\}$, one for each slot, such that for all i, j with $x_{ij} = 1$*

   $$pos_j v_i - p_j \geqslant pos_{j-1} v_i - p_{j-1}, \text{ and,}$$
   $$pos_j v_i - p_j \geqslant pos_{j+1} v_i - p_{j+1}$$

   In words, if bidder i is assigned slot j, he prefers that than a slot above or below – for the top-most slot, inequalities for slot below and for the bottom-most slot, inequalities for slot above holds.

   Prove that: an assignment x^* is allocatively efficient in the sponsored search auction model if and only if it is locally envy-free.

10. Consider an auction to sell m identical units of an object. Let there be n bidders with $n > m$. The valuations of the bidders for the object are $v_1, \ldots, v_n$, WLOG assume that $v_1 \geqslant \ldots \geqslant v_n$. Each bidder is interested in at most one unit of the object.

- What will be an efficient allocation of the items?
- What will be the VCG payment of the agents?
- Looking from a revenue (the sum payment of all the bidders) perspective do you see a problem with VCG?
- What alternative payment rule can you suggest that ensures a fraction $\lambda \in [0, 1]$ of the social welfare? [Hint: consider an appropriate Groves payment]