CS698W: Topics in Game Theory and Collective Choice

Problem Set 1
Course Homepage: `https://swaprava.wordpress.com/game-theory-collective-choice/`

1. Consider the game of Rock, Paper and Scissors played between two players. Each player in the game has three pure strategies *Rock*, *Paper* and *Scissor*. If two players choose same strategy, its a draw. Else, player who chooses rock is defeated by player who chooses paper. Player who chooses paper is defeated by player who chooses scissor and player who chooses scissor is defeated by player who chooses rock. Find mixed strategy Nash equilibria of this game.

2. Consider the following variation of the above game. Here player 1 suffers a minor handicap. If he wins with any other strategy apart from *paper*, the game results in a draw. Find mixed strategy Nash equilibria of this variation.

3. Find the mixed strategy Nash equilibria of a three-player game, in which each player has two actions: Action A and Action B. If each player chooses action A, they all gets payoff of 1. If each player chooses action B, they all get payoff of 4. For all other choices, the payoff is 0 for all the players.

4. Each individual of a group of hunters has two options: she may remain attentive to the pursuit of a stag, or catch a hare. If all hunters pursue the stag, they catch it and share it equally; if any hunter devotes her energy to catching a hare, the stag escapes, and the hare belongs to the defecting hunter alone. Each hunter prefers a share of the stag to a hare. If multiple hunters chase a hare, they share it. Find the Nash equilibria of the strategic game that models this situation. [You may place numbers appropriately to represent this scenario]

5. Consider two variants of the n-hunter stag hunt in which only m hunters, with $2 \leqslant m < n$, need to pursue the stag in order to catch it. Continue to assume that there is a single stag. Assume that a captured stag is shared only by the hunters that catch it.
   - Assume, as before, that each hunter prefers the fraction $1/n$ of the stag to a hare. Find the Nash equilibria of the strategic game that models this situation.
   - Assume that each hunter prefers the fraction $1/k$ of the stag to a hare, but prefers the hare to any smaller fraction of the stag, where k is an integer with $m \leq k \leq n$. Find the Nash equilibria of the strategic game that models this situation.

6. Extend the n-hunter stag hunt by giving each hunter K (a positive integer) units of effort, which she can allocate between pursuing the stag and catching hare. Denote the effort hunter i devotes to pursuing the stag by e_i, a nonnegative integer equal to at most K. The chance that the stag is caught depends on the smallest of all the hunters efforts, denoted $\min_j e_j$ (*"A chain is as strong as its weakest link"*). Hunter i's payoff to the action profile $(e_1, \ldots, e_n)$ is $2\min_j e_j - e_i$ (She is better off the more likely the stag is caught, and worse off the more effort she devotes to pursuing the stag, which means she catches fewer hares).

Is the action profile $(e, \ldots, e)$, in which every hunter devotes the same effort to pursuing the stag, a Nash equilibrium for any value of e? Is any action profile in which not all the players' effort levels are the same a Nash equilibrium?

7. Two animals are fighting over some prey. Each can be passive or aggressive. Each prefers to be aggressive if its opponent is passive, and passive if its opponent is aggressive; given its own stance, it prefers the outcome when its opponent is passive to that in which its opponent is aggressive. Formulate this situation as a strategic game and find its Nash equilibria. [You may place numbers appropriately to represent this scenario]

8. Each of n people chooses whether or not to contribute a fixed amount toward the provision of a public good. The good is provided if and only if at least k people contribute, where $2 \leqslant k \leqslant n$; if it is not provided, contributions are not refunded. Each person ranks outcomes from best to worst as follows:

   - any outcome in which the good is provided and she does not contribute
   - any outcome in which the good is provided and she contributes
   - any outcome in which the good is not provided and she does not contribute
   - any outcome in which the good is not provided and she contributes.

   Formulate this situation as a strategic game and find its Nash equilibria. [You may place numbers appropriately to represent this scenario]

9. Consider a normal form game $\langle N, (A_i)_{i \in N}, (u_i)_{i \in N} \rangle$ where $\exists \phi : A \mapsto \mathbb{R}$ such that for every player $i \in N$, for all $a_i, a_i' \in A_i$ and for all $a_{-i} \in A_{-i}$

$$u_i(a_i, a_{-i}) - u_i(a_i', a_{-i}) = \phi(a_i, a_{-i}) - \phi(a_i', a_{-i}).$$

   Prove that this game has a pure strategy Nash equilibrium.

10. **[Extensive-form].** Suppose that three players share a cake as follows: First, player 1 proposes a division (a division is a partition of the cake in three pieces – assume the cake is as simple as the interval $[0, 1]$), then players 2 and 3 simultaneously respond either "yes" or "no". If both say "yes" then the division is implemented: otherwise, no player receives anything. Each player prefers more of the cake to less.

    (a) Formulate as an extensive-form game with simultaneous moves. (Hint: a variation on the extensive-form in the class, with multiple players at any node. By formulate, you could simply give the game tree.)

    (b) Find the subgame perfect equilibria of the game. Also provide a Nash equilibrium that is not subgame perfect.

11. **[Altruistic players in the Neighboring Kingdoms' Dilemma].** In the classical Neighboring Kingdoms' Dilemma game, each of two players had two possible actions, agriculture (A) and defense (D). The monetary payoff to each action profile are given in the following table: Consider an altruistic variation of the game where each player not only cares about their own payoff, but also the other player's payoff. In particular, each player's modified

		Player 2: A	Player 2: D
Player 1	A	5,5	0,6
	D	6,0	1,1

payoff becomes his original payoff plus α times the original payoff of the other player. For example, player 1's modified payoff to action profile (A,A) is $5 + 5\alpha$ and payoff to action profile (A,D) is $0 + 6\alpha$.

(a) Write down the strategic form of this game for $\alpha = 1$. Is this game a Neighboring Kingdoms' Dilemma game (in terms of the conclusions about the outcome)? Explain your answer.

(b) Find the range of values of α or which the resulting game is the Neighboring Kingdoms' Dilemma. For values of α for which the game is not the Neighboring Kingdoms' Dilemma, find its Nash equilibria.

12. Two people are engaged in a joint project. If each person i puts in the effort $x_i \in [0, 1]$, which costs her $c(x_i)$, the outcome of the project is worth $f(x_1, x_2)$. The worth of the project is split equally between the two people, regardless of their effort levels.

    (a) Formulate this situation as a strategic game.

    (b) Find its Nash equilibria when

    i. $f(x_1, x_2) = 3x_1x_2$, $c(x_i) = x_i^2$, $i = 1, 2$.

    ii. $f(x_1, x_2) = 4x_1x_2$, $c(x_i) = x_i$, $i = 1, 2$.

    (c) In each case, is there a pair of effort levels that yields both players higher payoffs than the Nash equilibrium effort?

13. Describe the following situation as an extensive form game. Three piles of matches are on the table. First pile has only one match, second pile has two, third pile has three matches. Two players alternately remove matches from the table. In each move, the player whose turn it is to act can remove match from one and only one pile and must remove at least one match. The player removing the last match loses the game.

    By drawing the game tree, identify a strategy of one of the players by which (s)he can guarantee victory.

14. **[An investment race].** Two investors are involved in a competition with a prize of \$1. Each investor can spend any amount in the interval $[0, 1]$. The winner is the investor who spends the most; in the event of a tie each investor receives \$0.50. Formulate this situation as a normal form game and find its mixed strategy Nash equilibria.

15. **[Guess the average].** Each of n people announces a number in the set $\{1, \ldots, K\}$. A prize of \$1 is split equally between all the people whose number is closest to $\frac{2}{3}$ of the average number. Show that the game has a unique mixed strategy Nash equilibrium, in which each player's strategy is pure.

16. **[Air strike].** Army A has a single plane with which it can strike one of three possible targets. Army B has one anti-aircraft gun that can be assigned to defend one of these targets. The value of target k is v_k, with $v_1 > v_2 > v_3 > 0$. Army A can destroy a target only if the target is undefended and A attacks it. Army A wishes to *maximize* the expected value of the damage and army B wishes to *minimize* it. Formulate the situation as a (strictly competitive) normal form game and find its mixed strategy Nash equilibria.

17. **[Guessing right].** Players 1 and 2 each choose a member of the set $\{1, \ldots, K\}$. If the players choose the same number then player 2 pays \$1 to player 1; otherwise no payment is made. Each player maximizes his expected monetary payoff. Find the mixed strategy Nash equilibria of this (strictly competitive) game.

18. Formulate the following parlor game as an extensive game with imperfect information. First player 1 receives a card that is either H or L with equal probabilities. Player 2 does not see the card. Player 1 may announce that her card is L, in which case she must pay \$1 to player 2, or may claim that her card is H, in which case player 2 may choose to concede or to insist on seeing player 1's card. If player 2 concedes then he must pay \$1 to player 1. If he insists on seeing player 1's card then player 1 must pay him \$4 if her card is L and he must pay her \$4 if her card is H. Find the Nash equilibria of this game.

19. **[Bayesian game].** Consider two agents. Agent 1 is the seller of one indivisible item and agent 2 is the buyer. The type of player 1, θ_1 is interpreted as the minimum price at which he is willing to sell the item. Type of player 2, θ_2 is the maximum price at which she is willing to buy the item. Assume $\Theta_1 = \Theta_2 = [0, 1]$. The belief of each player, i.e., $f(\theta_{-i}|\theta_i)$, is the uniform distribution over $[0, 1]$. The protocol is the following. The seller and the buyer submits their bids, b_i, $i = 1, 2$. Trade happens at a price $\frac{b_1+b_2}{2}$ only if $b_2 \geqslant b_1$ and the payoffs are quasi-linear, i.e., $p - \theta_1$ for seller and $\theta_2 - p$ if p is the price. If $b_2 < b_1$, trade does not happen and both agents get zero payoff. Formulate this as a Bayesian game and find a Bayesian equilibrium.

20. **[Ice cream game].** Suppose consumers of a town are uniformly distributed over $[0, 1]$. Two ice-cream vendors are deciding which spot to locate their shops. Assume that the vendors' utilities are equal to the fraction of population they get. Every customer comes to her closest ice-cream shop – in case the shops are equidistant, she picks any shop uniformly at random. A customer's cost is the distance from his location to the nearest shop, and she always tries to minimize this cost. Considering this to be a strategic form game between the two vendors, where do you think they should locate their shops in a Nash equilibrium? If you were the planner of the town and minimizing the sum of the costs (we will call this the *social cost*) of the customers was your priority, where would you place the shops? What is the ratio of the social cost of the Nash equilibrium solution and your solution? What is this ratio when the number of vendors increase? Do you have a solution to minimize this number (e.g., by putting rents for different spots in the town)? If so, how should you price the spots?