CS698W: Topics in Game Theory and Collective Choice

Midterm – Semester 1, 2017-18.
Computer Science and Engineering
Indian Institute of Technology Kanpur
Total Points: 60, Time: 2 hours
ATTEMPT ALL QUESTIONS

1. Find the mixed strategy Nash equilibria of a three-player game, in which each player has two actions: Action A and Action B. If each player chooses action A, they all gets payoff of 1. If each player chooses action B, they all get payoff of 4. For all other choices, the payoff is 0 for all the players. **10 points.**

2. Suppose that two players share a cake as follows: First, player 1 proposes a division (assume the cake is as simple as the interval $[0, 1]$, hence a partition is a marker position $x \in [0, 1]$ such that $[0, x]$ belongs to player 1, and $(x, 1]$ belongs to player 2), then player 2 responds either "yes" or "no". If she say "yes" then the division is implemented: otherwise, no player receives anything. Each player prefers more of the cake to less.

   (a) Formulate this setting as an extensive-form game with appropriate definitions of the player actions and utilities.

   (b) Find subgame perfect equilibria of the game. Also provide a Nash equilibrium that is not subgame perfect – explain why this is *not* subgame perfect.

   5 + 5 points.

3. There are n departments in IIT Kanpur, and each of them is trying to convince the institute to sanction a total budget for the departments. If h_i is the hours (assume this is a nonnegative real number) put in by department i to prepare a proposal that costs them $c_i = w_i h_i^2$ (which accounts for the human efforts, machinery, electricity, printing etc. together), where $w_i > 0$. When the hours spent by the departments are given by the vector $(h_1, h_2, \ldots, h_n)$, the total budget allocated by the institute is

   $$\alpha \sum_{i=1}^{n} h_i + \beta \prod_{i=1}^{n} h_i, \quad \text{where } \alpha > 0, \beta \geqslant 0.$$

   Assume that the sanctioned budget is equally divided among the departments. Consider the game where every department chooses their hours to spend and payoffs are given by the budget allocated to them minus the cost. What restriction is required for the parameter β for this game to have a dominant strategy equilibrium? Find the DSE under that condition. **10 points.**

4. *Prove or disprove:* Borda and veto social welfare functions satisfy *independence of irrelevant alternatives (IIA)*. To prove, show this formally. To disprove, provide counterexamples. Explain your answer.

   Recall: Borda and veto are special cases of scoring rules $(s_1, \ldots, s_m)$ such that each alternative is weighted with the scores according to the alternatives' ranking for an agent and the ranking w.r.t. the sum of these scores becomes the social ranking. Borda scores: $(m-1, m-2, \ldots, 1, 0)$, veto scores: $(1, 1, \ldots, 1, 0)$. Independence of irrelevant alternative says if the relative ranks of a pair of alternatives *agree* in two profiles, the social rankings must agree in the two profiles.

5 + 5 points.

5. Consider a setting with n agents, where n is odd, and three facilities $\{a, b, c\} =: A$. Assume the locations of the facilities on a real line have a linear order $\leqslant$ such that $a < b < c$. Let the preferences P_i of every agent i belong to $\mathcal{P}^{\text{SP}} := \{P : P \text{ is a strict preference and single peaked w.r.t. } \leqslant\}$. Consider the *pairwise majority* social welfare function $F^{\text{Maj}} : \mathcal{P}^{\text{SP},n} \mapsto \mathcal{P}^{\text{SP}}$ which, $\forall\, a, b \in A$, ranks $aF^{\text{Maj}}(P)b$ if

$$|\{i : aP_i^{\text{SP}}b\}| > |\{i : bP_i^{\text{SP}}a\}|.$$

That is, F^{Maj} reflects the majority ranking between every pair of alternatives.

(a) Prove that F^{Maj} returns a well-defined ordering, i.e., it is complete and transitive. To show transitivity, one needs to show that there cannot be a case where $\exists P \in \mathcal{P}^{\text{SP},n}$ such that $aF^{\text{Maj}}(P)b$ and $bF^{\text{Maj}}(P)c$ and $cF^{\text{Maj}}(P)a$ – pairwise majority leading to a cycle. Such a social welfare function is called *Condorcet consistent*. This also shows that a *Condorcet winner* (an alternative that is undefeated by every other alternative in pairwise majority) exists for single peaked preferences. [*Hint*: consider a proof by contradiction]

(b) Is F^{Maj} single peaked? Argue why.

Recall: A preference relation P is single peaked w.r.t. an ordering $\leqslant$ of the alternatives if there exists an alternative x_P such that:

$$\begin{aligned} &\text{if } y < z \leqslant x_p \text{ then } zPy \\ &\text{if } x_p \leqslant z < y \text{ then } zPy. \end{aligned}$$

10 + 10 points.

Good Luck!