CS698W: Topics in Game Theory and Collective Choice

Midterm – Semester 1, 2017-18.
Computer Science and Engineering
Indian Institute of Technology Kanpur
Solution to selected problem(s)

5. Consider a setting with n agents, where n is odd, and three facilities $\{a, b, c\} =: A$. Assume the locations of the facilities on a real line have a linear order $\leqslant$ such that $a < b < c$. Let the preferences P_i of every agent i belong to $\mathcal{P}^{\text{SP}} := \{P : P$ is a strict preference and single peaked w.r.t. $\leqslant\}$. Consider the *pairwise majority* social welfare function $F^{\text{Maj}} : \mathcal{P}^{\text{SP},n} \mapsto \mathcal{P}^{\text{SP}}$ which, $\forall\ a, b \in A$, ranks $aF^{\text{Maj}}(P)b$ if

$$|\{i : aP_i^{\text{SP}}b\}| > |\{i : bP_i^{\text{SP}}a\}|.$$

That is, F^{Maj} reflects the majority ranking between every pair of alternatives.

(a) Prove that F^{Maj} returns a well-defined ordering, i.e., it is complete and transitive. To show transitivity, one needs to show that there cannot be a case where $\exists P \in \mathcal{P}^{\text{SP},n}$ such that $aF^{\text{Maj}}(P)b$ and $bF^{\text{Maj}}(P)c$ and $cF^{\text{Maj}}(P)a$ – pairwise majority leading to a cycle. Such a social welfare function is called *Condorcet consistent*. This also shows that a *Condorcet winner* (an alternative that is undefeated by every other alternative in pairwise majority) exists for single peaked preferences. [*Hint*: consider a proof by contradiction]

(b) Is F^{Maj} single peaked? Argue why.

Recall: A preference relation P is single peaked w.r.t. an ordering $\leqslant$ of the alternatives if there exists an alternative x_P such that:

$$\text{if } y < z \leqslant x_p \text{ then } zPy$$
$$\text{if } x_p \leqslant z < y \text{ then } zPy.$$

10 + 10 points.

A solution:

Part 1: The completeness of F^{Maj} is immediate since there are odd number of agents, for every $P \in \mathcal{P}^{\text{SP},n}$ and every pair $a, b \in A$ either $aF^{\text{Maj}}(P)b$ or $bF^{\text{Maj}}(P)a$ but not both.

We show transitivity as follows. Since the domain of the social choice function consists of preferences that are strict orderings and single peaked w.r.t. $\leqslant$, there are fewer possible preferences in the set $\mathcal{P}^{\text{SP}}$ given by (we denote $\succ$ to denote a general preference ordering in this domain):

$$\begin{array}{ll} a \succ b \succ c & \text{group 1} \\ b \succ a \succ c & \text{group 2} \\ b \succ c \succ a & \text{group 3} \\ c \succ b \succ a & \text{group 4} \end{array} \tag{1}$$

Suppose for contradiction F^{Maj} is not transitive. Hence at least one of the cases below must hold where F^{Maj} returns a cycle.

Case 1: Suppose $\exists\ P \in \mathcal{P}^{\text{SP},n},$ s.t. $a\ F^{\text{Maj}}(P)\ b\ F^{\text{Maj}}(P)\ c\ F^{\text{Maj}}(P)\ a$: Call the number of agents in group i in Eq. 1 by n_i, $i = 1, 2, 3, 4$. Since $c\ F^{\text{Maj}}(P)\ a$, it must be the case that $n_3 + n_4 \geqslant \frac{n+1}{2}$, and hence $n_1 + n_2 \leqslant \frac{n-1}{2}$. But since $a\ F^{\text{Maj}}(P)\ b$ as well, it must be the case that $n_1 \geqslant \frac{n+1}{2}$. But this is a contradiction. Hence this case cannot occur.

Case 2: Suppose $\exists\ P \in \mathcal{P}^{\text{SP},n},$ s.t. $b\ F^{\text{Maj}}(P)\ a\ F^{\text{Maj}}(P)\ c\ F^{\text{Maj}}(P)\ b$: Since $a\ F^{\text{Maj}}(P)\ c$, it implies $n_3 + n_4 \leqslant \frac{n-1}{2}$. But since $c\ F^{\text{Maj}}(P)\ b$, $n_4 \geqslant \frac{n+1}{2}$, which is a contradiction.

Case 3: Suppose $\exists\ P \in \mathcal{P}^{\text{SP},n},$ s.t. $a\ F^{\text{Maj}}(P)\ c\ F^{\text{Maj}}(P)\ b\ F^{\text{Maj}}(P)\ a$: equivalent to *Case 2*.

Case 4: Suppose $\exists\ P \in \mathcal{P}^{\text{SP},n},$ s.t. $b\ F^{\text{Maj}}(P)\ c\ F^{\text{Maj}}(P)\ a\ F^{\text{Maj}}(P)\ b$: equivalent to *Case 1*.

Case 5: Suppose $\exists\ P \in \mathcal{P}^{\text{SP},n},$ s.t. $c\ F^{\text{Maj}}(P)\ a\ F^{\text{Maj}}(P)\ b\ F^{\text{Maj}}(P)\ c$: equivalent to *Case 1*.

Case 6: Suppose $\exists\ P \in \mathcal{P}^{\text{SP},n},$ s.t. $c\ F^{\text{Maj}}(P)\ b\ F^{\text{Maj}}(P)\ a\ F^{\text{Maj}}(P)\ c$: equivalent to *Case 2*.

Part 2: Suppose F^{Maj} is not single peaked. We consider the case where $a\ F^{\text{Maj}}(P)\ c\ F^{\text{Maj}}(P)\ b$ for some P (the other case, $c\ F^{\text{Maj}}(P)\ a\ F^{\text{Maj}}(P)\ b$ for some P is symmetric).

Since $a\ F^{\text{Maj}}(P)\ c$, then at least $\frac{n+1}{2}$ agents place a above c. As the preferences are *single peaked* with $a < b < c$, it implies that those $\frac{n+1}{2}$ agents place b above c as well. But that contradicts the fact that $c\ F^{\text{Maj}}(P)\ b$. Hence F^{Maj} must be single peaked.