CS698W: Topics in Game Theory and Collective Choice

Endsem – Semester 1, 2017-18.
Computer Science and Engineering
Indian Institute of Technology Kanpur
Total Points: 30, Time: 2 hours
ATTEMPT ALL QUESTIONS

1. A seller is selling an object to an agent whose value (type) for the object lies in the interval $I \equiv [0,1]$. The seller uses an allocation rule $f : I \mapsto [0,1]$ and a payment rule $p : I \mapsto \mathbb{R}$. Denote the mechanism (f,p) as M.

   Fix an $\epsilon \in (0,1]$. The mechanism M satisfies only **local** incentive compatibility constraints: for every $t, s \in I$ such that $|s - t| \leqslant \epsilon$, we have

   $$tf(t) - p(t) \geqslant tf(s) - p(s).$$

   Show that M is (global) dominant strategy incentive compatible. Note that this requires that both the allocation and the payment rule should satisfy the DSIC condition for any arbitrary t and t' in that interval. Hint: use Myerson's result for the allocation, but for the payment, deriving from first principle of DSIC may be useful. **5 + 5 points.**

   Solution: First show that the allocation is non-decreasing. Consider any $t, s \in I$ such that $|s - t| \leqslant \epsilon$. Using the local IC condition, we know

   $$\begin{aligned} tf(t) - p(t) &\geqslant tf(s) - p(s), \text{ and} \\ sf(s) - p(s) &\geqslant sf(t) - p(t) \end{aligned}$$

   Adding the inequalities, we get,

   $$(t - s)(f(t) - f(s)) \geqslant 0.$$

   For any arbitrary t and t' in I the same conclusion can be reached by a finite number of such pairs of points that are within ϵ of each other. This proves that the allocation function is non-decreasing.

   Now, for the payment part, we show that the rule is DSIC by considering two profiles $t, t' \in I$ which are arbitrary (not necessarily within ϵ of each other). WLOG, assume $t < t'$ – a similar argument works if $t > t'$. Consider a sequence of types $t = t_0 < t_1 < t_2 < \ldots < t_m = t'$ such that $|t_i - t_{i+1}| < \epsilon, \forall i = 0, \ldots, m-1$. Such a finite m is guaranteed to exist since $\epsilon > 0$ and $|t - t'| \leqslant 1$. In particular, we need an m s.t. $m \geqslant \frac{|t-t'|}{\epsilon}$. Hence,

   $$\begin{aligned} t_0 f(t_0) - p(t_0) &\geqslant t_0 f(t_1) - p(t_1) \\ t_1 f(t_1) - p(t_1) &\geqslant t_1 f(t_2) - p(t_2) \\ &\vdots \\ t_{m-1} f(t_{m-1}) - p(t_{m-1}) &\geqslant t_{m-1} f(t_m) - p(t_m) \end{aligned}$$

Adding and reorganizing, we get

$$\begin{aligned} t_0 f(t_0) - p(t_0) &\geqslant (t_0 - t_1) f(t_1) + (t_1 - t_2) f(t_2) + \ldots + (t_{m-2} - t_{m-1}) f(t_{m-1}) \\ &\quad + (t_{m-1} - t_0) f(t_m) + t_0 f(t_m) - p(t_m) \\ &\geqslant t_0 f(t_m) - p(t_m) \end{aligned}$$

The second inequality holds since f is non-decreasing and $t_0 < t_1 < t_2 < \ldots < t_m$, which gives $(t_i - t_{i+1}) f(t_{i+1}) \geqslant (t_i - t_{i+1}) f(t_m), \forall i = 0, \ldots, m-1$. Hence proved. ■

2. Suppose a seller is selling a single object to a single agent whose value is distributed in $[0, 1]$ using a distribution G with density g. The density is positive at every point in its domain.

   (a) Write down the expression for expected payment of an incentive compatible and individually rational mechanism (f, p) in terms of virtual value of the agent.

   Solution: This was derived in class

   $$\int_0^1 w(t) g(t) f(t) dt.$$

   Where

   $$w(t) = \left(t - \frac{1 - G(t)}{g(t)} \right).$$

   ■

   (b) Find the optimal mechanism if the function $H(x) = xG(x)$ for all $x \in [0, 1]$ is strictly convex.

   Solution: Since $H(x)$ is strictly convex, $H'(x)$ is strictly increasing. Expected payment is given by

   $$\int_0^1 (tg(t) + G(t) - 1) f(t) dt.$$

   Consider the expression $(tg(t) + G(t) - 1) =: y(t)$. Clearly $y(t)$ is strictly increasing and $y(0) = -1$ and $y(1) = g(1) > 0$. Hence, there exists a unique x^* such that $y(x^*) = 0$ and for all $t < x^*, y(t) < 0$ and for all $t > x^*, y(t) > 0$. Therefore, the optimal mechanism is to set

   $$f(t) = \begin{cases} 1 & \text{if } t > x^* \\ \alpha \in [0, 1] & \text{if } t = x^* \\ 0 & \text{if } t < x^* \end{cases}$$

   Payment is $p(t) = f(t) x^*$ as given by Myerson's result. ■

4 + 6 points.

3. Consider a sports TV channel which is selling the advertisement time of 20 seconds between consecutive overs of a cricket match. There are four advertisers $\{1, 2, 3, 4\}$. The length of the

advertisement of advertiser i is denoted as L_i and is given by $L_1 = 12, L_2 = 7, L_3 = 7, L_4 = 10$. The advertisement lengths are publicly known. If an advertisement is displayed, then it has to be displayed entirely and the total length of all the advertisements chosen to be displayed must be less than or equal to 20 seconds (finishing the ads early is admissible but overshooting is not). Also, order of the ads does not matter to the advertisers.

Advertiser i gets a value $v_i \in \mathbb{R}_{\geqslant 0}$ if its advertisement is displayed, which is his private information. The values are given as follows: $v_1 = 6, v_2 = 12, v_3 = 7, v_4 = 8$.

(a) What are the allocations and payments in the pivotal (VCG) mechanism at this profile?

Solution: The allocation $\{2, 4\}$ maximizes the social welfare (gives 20) among the feasible allocations. It is easy to check that the payments are: $p_1 = 0, p_2 = 15-8 = 7, p_3 = 0, p_4 = 19 - 12 = 7$. ■

(b) The TV channel is considering to use the following (**greedy**) allocation rule. For every advertiser i, it computes a real number $\kappa_i := \hat{v}_i / L_i$, where $\hat{v}_i$ is the reported value of agent i. The TV channel then orders the advertisers in decreasing values of κ_i. Starting from the advertiser with the highest κ_i value, the TV channel chooses advertisers with top k values of κ_i whose total length of advertisements is less than or equal to 20 seconds and adding the $(k+1)$-th advertiser requires more than 20 seconds of length.

i. State why this allocation rule is implementable in dominant strategies. You may use Myerson's characterization lemma for single-parameter domains but explain fully how the conditions hold in this case.

Solution: This is a single-parameter domain and hence Myerson's characterization result holds. The allocation is clearly **non-decreasing** as an increase in the valuation v_i can only increase κ_i and hence the chance of an advertiser i's ad being shown. By Myerson's result, an increasing allocation rule is implementable. ■

ii. Compute the payments that implements this rule in dominant strategies for the same valuations as mentioned before.

Solution: On the valuation profile $v_1 = 6, v_2 = 12, v_3 = 7, v_4 = 8$, and the length of the ads, the allocation is $\{2, 3\}$. The payment that implements it is given by

$$\begin{aligned} p_2(v) &= 12 \cdot 1 - \int_0^{12} f_2(x_2, v_{-2}) dx_2 \\ &= 12 \cdot 1 - \int_{5.6}^{12} 1 \cdot dx_2 \\ &= 5.6 \end{aligned}$$

The first equality is due to the payment formula of Myerson's result with the constant term being zero. The second equality comes because the value of agent 2 has to cross the threshold so that it is considered as the winning advertiser. The ratio κ_i's are $0.5, 12/7, 1, 0.8$ for the agents, and the value of agent 2 needs to be at least 5.6 to become the second winner of the advertisement slot (along with agent 3). Similarly, we can find that the payment of 3 is $p_3(v) = 5.6$. ■

5 + (2 + 3) points.

Theorem 1 (Myerson 1981) *Suppose $T_i = [0, b_i], \forall\ i \in N$ and the valuations are in product form. An allocation rule $f : T \mapsto \Delta A$ and a payment rule $(p_1, p_2, ..., p_n)$ is DSIC iff*

1. *The allocation f is non-decreasing, and,*
2. *Payment is given by*

$$p_i(t_i, t_{-i}) = p_i(0, t_{-i}) + t_i f_i(t_i, t_{-i}) - \int_0^{t_i} f_i(x_i, t_{-i})\ dx_i.$$

Good Luck!