CS698A: Selected Topics in Mechanism Design **Jan-Apr 2018**

Lecture 15: Miscellaneous Topics

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Disclaimer: *These notes aggregate content from several texts and have not been subjected to the usual scrutiny deserved by formal publications. If you find errors, please bring to the notice of the Instructor.*

15.1 Strategic aspects of DA algorithm

15.1.1 Truthfulness

Consider the men-proposing version of the DA algorithm with preference of men and women as follows:

P_{m_1}	P_{m_2}	P_{m_3}	P_{w_1}	P_{w_2}	P_{w_3}
w_2	w_1	w_1	m_1	m_3	m_1
w_1	w_3	w_2	m_3	m_1	m_3
w_3	w_2	w_3	m_2	m_2	m_2

The DA algorithm proceeds in the following manner:

Step 1: $m_1 \to w_2$, $m_2 \to w_1$, $m_3 \to w_1$
At the end of step 1, m_1 is matched to w_2 and m_3 is matched to w_1 while m_2 is rejected.

Step 2: $m_2 \to w_3$ and w_3 accepts the proposal and the algorithm terminates.

Now, suppose w_1 misreports her preferences as $m_1 > m_2 > m_3$, then the DA algorithm results in the matching $\{(m_1, w_1), (m_2, w_3), (m_3, w_2)\}$ which is different from the matching in the above case. Thus, we conclude that the men(women)-proposing DA algorithm is not **truthful** for women(men).

Theorem 15.1. *The men(women)-proposing DA algorithm is strategy proof for men(women)*

At this point, one might ask the question - "Can there be an algorithm that is truthful for both?", the answer to which is NO.

Theorem 15.2. *No stable matching algorithm can be strategy proof for both men and women*

15.1.2 Open Research directions in matching

- Fairness consideration: Is there a stable match that is more egalitarian for both men and women
- Matching under multiple attributes: take for example the case of kidney exchange. It is a kind of house allocation problem but not anyone can be allotted any kidney. The donor and receiver should have compatible blood groups etc.

  Similarly, if preferences are multidimensional. Example in case of student selecting a university student can select university based on several attributes like distance from his home, international ranking, weather etc.
- Monetary Transfer: Classical quasi-linear setting. In this domain, questions related to money can be asked.

15.2 Strategic Network Formation

In the most basic sense, a network is any collection of objects in which some pairs of these objects are connected by links. This definition is very flexible: depending on the setting, many different forms of relationships or connections can be used to define links In the current topic, we ask for the incentives for individuals to form links and the reason for which sort of network may result due to their strategic choices.

The game theoretic model of network formation :

- takes into account the costs and benefits for agents associated with networks.
- The agents are nodes and they choose links. Some examples of strategic networks are:
  - countries with trade relations
  - people choosing friends
  - researchers with research collaborations
  - employers with companies, etc.
- contrasts incentives of individual to form a relationship with what might be best for society(social efficiency)

Once proceed in this direction, we have all kinds of modelling choices to make. How should we model incentives to form and sever links?

To answer the above question we need to answer several questions related to the strtegic networks which are

- Is consensus needed?
  - making friend requires consensus of both the parties.
  - in citation networks, consensus is not needed to cite a researcher.
- Can people coordinate changes in network?

  Example: A will form an alliance with B iff they form an alliance with C
- Is the process static or dynamic,that is, are all the agents coming in together and forming all the links at the same time or are the links formed eventually?
- How sophisticated are the agents, i.e., are the agents calculating the value of their alliance and then forming one or are they just randomly forming the alliance?
- Are the links adjustable in intensity?
- can they compensate each other for relationship?

Some questions that one should keep in mind while going through this topic are

- Which networks are likely to form?
- Are some networks more stable than the others to various perturbations?
- Are networks efficient from the societies point of view?
- How inefficient they are if not efficient?
- Can intervention help improve efficiency?

15.2.1 The Connections Model: Jackson-Wollinsky,1996

The model that we are going to take into consideration was given by Jackson-Wollinsky in 1996. The various attributes of a network g in this model are:

$$
\begin{array}{rcl}
u_i(g) & : & \text{payoff to agent i in network } g \\
0 \leq \delta_{ij} \leq 1 & : & \text{a benefit parameter for connection between i and j} \\
0 \leq c_{ij} & : & \text{cost to i to maintain a link to j} \\
l_{ij} & : & \text{shortest path length between i and j} \\
u_i(g) & = & \sum_{j \in N \setminus \{i\}} \delta_{ij}^{l_{ij}} - \sum_{j \in N_i(g)} c_{ij}, \text{ where } N_i(g) \text{ is the neighbour set of player i}
\end{array}
$$

Example in symmetric case:

Consider a case where $\delta_{ij} = \delta$ and $c_{ij} = c \ \forall \ ij$. The line connecting the nodes represents a link. The numbers written outside the nodes are the utilities of the players.

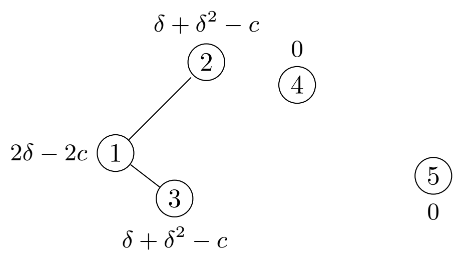

Figure 15.1: Strategic Network and corresponding utilities of the players: Example 1

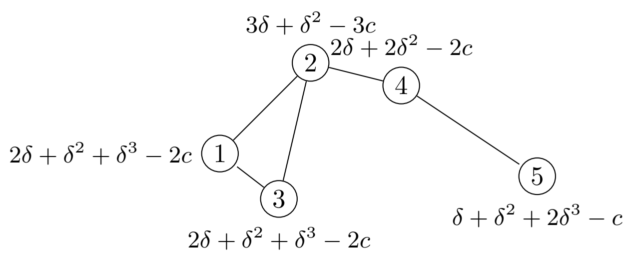

Figure 15.2: Strategic Network and corresponding utilities of the players: Example 2

Modeling Incentives

Consider a world where we need consensus to form links. So, two individuals have to agree to be friends, and one can't form a friendship with somebody else if they don't both agree to be friends. Now one way we might think of modeling this would be as just modeling it as a game. And the simplest possible game one can imagine is everybody just announces who they want to be friends with. So in case of two players, if both announce each other, then we form a friendship between them, and if they don't both announce each other, then we don't form a friendship. Now, we will see what Nash equilibrium suggests in this type of setting.

As stated before, consider a case with two players. If both announce to have friendship link is formed and both get a payoff 1, else both get 0. The situation is illustrated below.

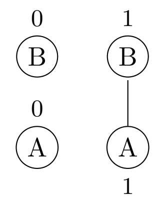

Figure 15.3: Two outcomes of the game

One can clearly see that in this case both these outcomes qualify as Nash Equilibrium. This result is unsatisfactory as it says anything can happen. So, Nash Equilibrium is not very helpful in determining the type of network in Strategic Network formation.

15.2.1.1 Pairwise Stability

A graph is said to be pairwise stable if:

- No agent gains from severing a link - relationships must be beneficial to maintain. Mathematically,
  $\forall (ij) \in g$
  $u_i(g) \geq u_i(g \setminus (ij))$
  $u_j(g) \geq u_j(g \setminus (ij))$
- No two agents both gain from adding a link (at least one strictly) - beneficial relationships are pursued when available. That is
  $\forall (ij) \notin g$
  if $u_i(g + (ij)) \geq u_i(g)$ then $u_j(g + (ij)) \leq u_j(g)$

This is a weak concept because

- It only looks at pairs of individuals
- It only looks at one ink at a time

Examples:

1. Consider the case of Figure 15.3. As one can see clearly, the first graph is not Pairwise stable while the second graph is.
2. Consider the sequence of graphs given in Figure 15.4. Only the last graph is pairwise stable.

Note: The pairwise stable network gives worse payoff than some unstable ones.Consider for example the case of Figure 15.4. The last graph is pairwise stable but not Pareto Efficient.

15.2.1.2 Pareto Efficiency

A network g is PE if $\nexists$ g' such that

$$u_i(g') \geq u_i(g) \ \forall \text{ i} \in N \text{ and strict for some j} \in N$$

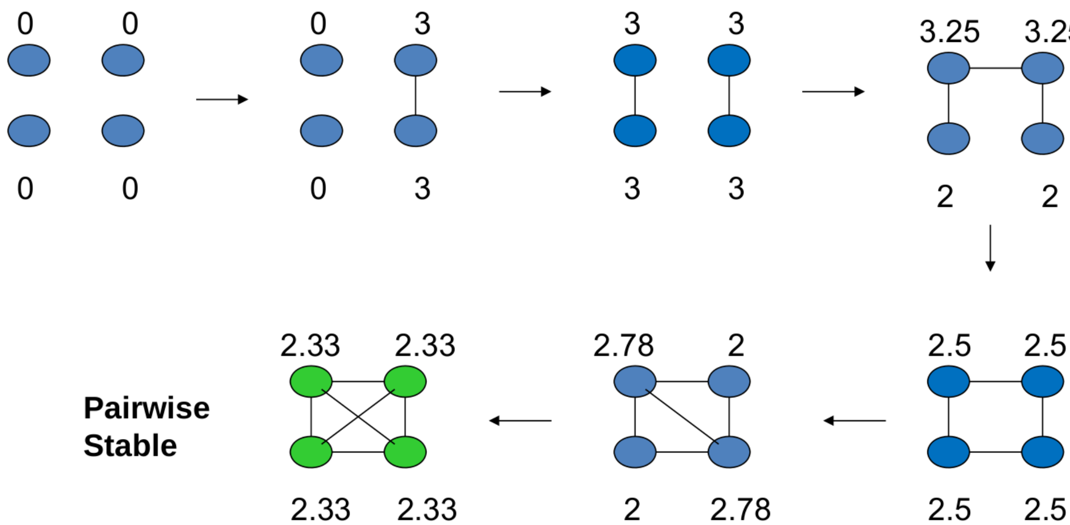

Figure 15.4: Step by step development of strategic network

15.2.1.3 Efficiency

A graph $g*$ is efficient if

$$g* \in \arg\max_{g \in \mathscr{G}} \sum_{i \in N} u_i(g)$$

Note: Efficiency implies PE.

Consider again the symmetric version of connection model with benefit parameter δ and cost for connecting two nodes c. We now present a theorem that tells us about the type of network that are efficient depending on the values of c.

Theorem 15.3. *(Low cost) If $c < \delta - \delta^2$; complete network is uniquely efficient*

(Medium cost) If $\delta - \delta^2 < c < \delta + (n-2)\delta^2/2$; star networks are uniquely efficient

(High cost) If $c > \delta + (n-2)\delta^2/2$; empty network is efficient

Proof. (Part 1) If ij are not connected then the benefit to each player is at most δ^2 but if they are directly connected then the benefit is $\delta - c$

It is better for the players to connect directly when

$$\delta - c > \delta^2 \Rightarrow c < \delta - \delta^2$$

(Part 2 and 3) Suppose $c > \delta - \delta^2$. Consider a collection of k players.

- Value of a star network with k players $= 2(k-1)(\delta - c) + (k-1)(k-2)\delta^2$
- Value using any other network with k players and m links (such that $m \geq k-1$) is at most $= 2m(\delta - c) + (2\binom{k}{2} - 2m)\delta^2$

Difference between the two values $= 2(m-(k-1))(\delta^2) - (\delta - c) > 0$ when $m > k$
Now suppose m=k-1 and network is not a star, then some pairs will be at a distance of more than 2, so the value of network will be less than a star. Therefore, star networks are better.

Now, one might ask if two stars are better than one and it turns out that the answer is no. The proof of this statement is left as an exercise to the reader.

It will be efficient to form a star network with n players till the total value of the network is greater than zero

$$\Rightarrow 2(n-1)(\delta - c) + (n-1)(n-2)\delta^2 > 0 \Rightarrow c < \delta + (n-2)\delta^2/2$$

Thus, when $c > \delta + (n-2)\delta^2/2$, the graph will be empty ■

15.2.1.4 Pairwise Stability

Strategic Networks in Low cost and High cost conditions as given by the above theorem are pairwise stable. In case of medium cost,

- (Medium low) If $\delta - \delta^2 < c < \delta$, Star is pairwise stable.
- (Medium high) If $\delta < c < \delta + (n-2)\delta^2/2$, Star is not pairwise stable.

15.3 Summary

We started this lecture by discussing the strategic aspects of DA algorithm. In course of lecture we observed that the men(women)-proposing DA algorithm is not truthful for women(men). We also saw that the men-proposing DA algorithm was strategy-proof for men and also that there is no algorithms that is strategy-proof for both men and women simultaneously. We also introduced the topic of formation of Strategic Networks, studied The Connections Model and familiarized with the notions of stability and efficiency in this setting.