CS698A: Selected Topics in Mechanism Design	**Jan-Apr 2018**

Lecture 14: Stability and Optimality of Deferred Acceptance Algorithm

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Disclaimer: *These notes aggregate content from several texts and have not been subjected to the usual scrutiny deserved by formal publications. If you find errors, please bring to the notice of the Instructor.*

14.1 Recap

In the last lecture we have studied about the *stability* at a profile where no pair of men and women can block an allocation/matching. And now we know that, the Deferred Acceptance algorithm ensures stability for both the versions, men-proposing and women-proposing. But now the question is, can we compare both these stable matches?

14.2 Compare among stable matchings

Definition 14.1 *A matching μ is men-optimal stable matching if μ is stable and for any other stable matching μ'. we have,*

$$\mu(m)P_m\mu'(m) \qquad or \qquad \mu(m) = \mu'(m)$$

Definition 14.2 *A matching μ is women-optimal stable matching if μ is stable and for any other stable matching μ'. we have,*

$$\mu(w)P_w\mu'(w) \qquad or \qquad \mu(w) = \mu'(w)$$

Remark: If there exist two men-optimal stable matching, then must differ for at least one man (in-fact for two man), and since preferences are strict hence this man must be worse off in one of these stable matches. Therefore, men(women)-optimal stable matching is unique.

Theorem 14.3 *The men(woman)-proposing Deferred Acceptance Algorithm terminates at a unique men(woman)-optimal stable matching.*
Proof:

Definition 14.4 *A woman w is possible for a man m if (m, w) is matched in same stable match.*

Claim 14.5 *A woman who is possible never rejects the man in this algorithm.*
Proof: *Clearly, true for round 1. Let m_1 and m both proposes w, but w prefers m_1 over m, and (m_1, w) is possible and (w, m) is impossible then, $\nexists$ any stable match having (m, w) otherwise, (m_1, w) will be a blocking pair.*
To prove for other rounds let us assume if claim holds till round n, say in round $n+1$, w rejects m, this implies that $\exists m'$ whose proposal is accepted by w in round $n + 1$. All woman who rejected m' must be impossible (by induction hypothesis) that means $\nexists$ any stable match (m', i) where, $\forall i \in W$ and i have rejected m' in rounds earlier than $n + 1$. We can conclude that w must be impossible for m, because if its not true or $\exists$ some stable match where (m, w) are matched then, there are only two cases:

(1.) m' had matched to woman above w in the preference order of m', but those women are impossible for m' as discussed above.
(2.) m' had matched to woman below w in the preference order of m', then (m', w) is a blocking pair. Thus the claim is proved. ∎

From the claim above it can concluded that DA algorithm is assigning the best possible woman to every man and hence it is $men - optimal$. ∎

14.3 More structures of stable matching

Some interesting questions are: Can both sides are happy? that is, does there exist a matching that is both side's optimal? The general answer is NO. We saw men and women optimal solutions to be different. But, something more is true. Let us explore the structure of the stable matchings a bit more.

Theorem 14.6 *Let μ and μ' be two stable matchings. Then, $\mu(m)P_m\mu'(m)$ or $\mu(m) = \mu'(m)$ $\forall m \in M$ if and only if $\mu'^{-1}(w)P_w\mu^{-1}$ or $\mu^{-1} = \mu'^{-1}$ $\forall w \in W$.*

Proof: *Let μ and μ' be two stable matchings with $\mu(m)P_m\mu'(m)$ $\forall m \in M$. Assume for contradiction that $\mu^{-1}(w)P_w\mu'^{-1}$ for some w. Suppose $m \equiv \mu^{-1}(w)$ therefore, $wP_m\mu'(m)$. Hence, (m,w) is a blocking pair of μ', contradicting the fact that μ' is stable.*
The proof for the other direction is symmetric. ∎

The previous two theorems say that the men-optimal stable mtching is the worst stable matching for woman and vice-versa.
We can define a binary relation between stable matching.

Definition 14.7 *We say $\mu \triangleright \mu'$ if for every $m \in M$, either $\mu(m)P_m\mu'(m)$ or $\mu(m) = \mu'(m)$*
[Equivalently, $\mu'^{-1}(w)P_w\ \mu^{-1}(w)$ or $\mu'^{-1}(w) = \mu^{-1}(w)$ $\forall w \in W$].

Note: $\triangleright$ is not a complete relation as can not compare all stable matchings. But an immediate corollary of the previous two theorems.

Corollary 14.8 *For any stable matching μ, $\mu^m \triangleright \mu \triangleright \mu^w$.*

For any pair of stable matchings μ, μ' define another matching
$\mu'' \equiv (\mu\ V^m\ \mu')$:

$$\mu''(m) = \begin{cases} \mu(m), & if \quad \mu(m)P_m\mu'(m) \quad\quad or \quad\quad \mu(m) = \mu'(m) \\ \mu'(m), & if \quad \mu'(m)P_m\mu(m) \end{cases}$$

We write this equivalently as for all $m \in M$,

$$(\mu V^m \mu')(m) = \max_{P_m}(\mu(m), \mu'(m)).$$

Similarly, we can define the matching $\mu V^w \mu'$ as follows: for every $w \in W$, we define

$$(\mu V^w \mu')^{-1}(w) = \max_{P_w}(\mu^{-1}(w), \mu'^{-1}(w)).$$

Theorem 14.9 *For every pair of stable matchings μ, μ', $(\mu\ V^m\ \mu')$ and $(\mu\ V^w\ \mu')$ is a stable matching.*

Proof: *The proof of this theorem will consist of two parts. The first part is to prove that $\mu V^m \mu'$ is a matching and in the second part we will prove that it is a stable matching.*

Part(1)

Take a pair of stable matchings μ, μ' and are shown below and let $\mu'' \equiv (\mu\ V^m\ \mu')$. Assume for contradiction that μ'' is not a matching. Then, there must exist $m, m' \in M$ such that $\mu''(m) = \mu''(m')$. Then, it must be the case that, there is some $w \in W$ with $w := \mu(m) = \mu'(m')$ and $wP_m\mu'(m)$ and $wP_{m'}\mu(m')$. There is also a similar case where the role of μ and μ' is reversed. Since μ' is stable, then $m'P_wm$. But, then (m', w) form a blocking pair of μ, contradicting the fact that μ is stable.

	µ''	µ	µ'
. . .			
m	w	w	µ'(m)
. . .			
m'	w	µ(m')	w
. .			

Part(2)

Here, we show that μ'' is a stable matching. Assume for contradiction (m, w) is a blocking pair of μ''. Suppose m is matched to w_1 and w is matched to m_1 in μ'' as shown below. Hence,

mP_wm_1*, and* wP_mw_1.

	µ''	µ	µ'
. . .			
m	w_1	w_1	w_2
. . .			
m_1	w		
. .			

Now, suppose $w_1 = \mu(m)$ and $w_2 = \mu'(m)$. It is possible that $w_1 = w_2$ and hence, $w_1P_mw_2$ or $w_1 = w_2$ in μ''. We can conclude that,

$$wP_mw_1 \text{ and } wP_mw_2.$$

Case(a) *We now argue that w is not matched to m_1 in μ. Suppose for contradiction, w is matched to m_1 in μ. Note that $\mu(m) = w_1$ as shown below. Since, $mP_w m_1$ and $wP_m w_1$, (m, w) form a blocking pair for μ.*

	μ"	μ	μ'
. . .			
m	w_1	w_1	w_2
. . .			
m_1	w	w	
. .			

Case(b) *Similarly, we argue that w is not matched to m_1 in μ'. Suppose for contradiction that, w is matched to m_1 in μ'. Note that $\mu'(m) = w_2$ as shown below. Since $mP_w m_1$ and $wP_m w_2$, (m, w) form a blocking pair for μ'.*

	μ"	μ	μ'
. . .			
m	w_1	w_1	w_2
. . .			
m_1	w		w
. .			

Since w is not matched to m_1 in both μ and μ', w and m_1 cannot be matched with each other in μ'' and therefore, (m, w) will not be a blocking pair. This leads to the contradiction.
A similar proof will work to show that $(\mu \ V^w \ \mu')$ is a stable matching. ■

Exercise: To prove the above theorem for $\mu'' \equiv (\mu \ \Lambda^m \ \mu')$:

$$\mu''(m) = \begin{cases} \mu'(m), & if \quad \mu(m) P_m \mu'(m) \quad or \quad \mu(m) = \mu'(m) \\ \mu(m), & if \quad \mu'(m) P_m \mu(m) \end{cases}$$

References

[Book] DEBASIS MISHRA , "Theory of Mechanism Design," ECONOMICS AND PLANNING UNIT, INDIAN STATISTICAL INSTITUTE